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A Review of the Hierarchy of Plithogenic, Neutrosophic, and Fuzzy Graphs: Survey and Applications

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Abstract: As many readers may know, graph theory is a fundamental branch of mathematics that examines networks consisting of nodes and edges, with a focus on their paths, structures, and properties [157]. A Fuzzy Graph extends this concept by assigning a membership degree between 0 and 1 to each edge and vertex, capturing the level of uncertainty. Expanding on this idea, the Turiyam Neutrosophic Graph was introduced as an extension of both Neutrosophic and Fuzzy Graphs. Plithogenic graphs, in turn, offer a powerful approach for managing uncertainty. In this paper, we explore the relationships among various graph classes, including Plithogenic graphs, and investigate other related graph structures.

Keywords: Neutrosophic graph, plithogenic graphs, Turiyam Neutrosophic graph, Fuzzy graph

1. Introduction

1.1 Graph Theory

As the readers are aware, graph theory, a fundamental branch of mathematics, examines networks composed of nodes and edges, focusing on their paths, structures, and properties [157]. It is well-known that graph theory has been extensively studied for a wide range of applications[70, 98, 148, 172, 197, 350, 461]. In graph theory, various graph classes, structures, and algorithms have been explored, such as those related to tree structures[133, 163], path structures[107], and linear layouts[142, 164], each tailored to specific objectives. It is also known that focusing on specific structures rather than arbitrary ones allows for the development of more efficient algorithms, illustrating the practical benefits of using these graph classes[105].

1.2 Fuzzy, Neutrosophic, Turiyam neutrosophic, and Plithogenic Graphs

To address the uncertain parameters and concepts in the world, graphs such as Fuzzy, Neutrosophic, Turiyam Neutrosophic graphs, and Plithogenic Graphs are being studied.

A Fuzzy Graph is a graph in which each edge and vertex is assigned a membership degree between 0 and 1, reflecting the level of uncertainty. To put it simply, a Fuzzy Graph is a graphical representation of a fuzzy set (cf. [162, 278, 477, 478]). In real-life applications, fuzzy graphs are used in areas like social networks, decision-making, and transportation systems to model imprecise relationships or uncertain connections[318, 377]. Due to their applicability in various fields, they have been the subject of extensive research. Within fuzzy graph theory, various graph classes have been proposed to generalize fuzzy graphs or adapt them for real-world applications. These include Intuitionistic Fuzzy Graphs [341], Bipolar Fuzzy Graphs [15], Fuzzy Planar Graphs [392], Irregular Bipolar Fuzzy Graphs [391], General Fuzzy graph[173, 336], Semi Fuzzy Graph[52], and Complex Hesitant Fuzzy Graphs [6], among others. Studying these classes helps researchers uncover common properties, develop specialized algorithms, and apply findings to practical problems.

To address uncertainty and the relationships between concepts, several graph classes have been introduced, including Fuzzy Graphs [318, 377], Vague Graphs [102, 369, 393], Plithogenic graphs [183, 411, 425, 426], weighted graphs[313, 459, 466], probabilistic graphs[171, 264, 303], Vague hypergraphs [25], Graph entropy (cf.Fuzzy entropy[265]) [144], N-graphs[14], N-hypergraphs[19], Markov graphs[266], Hyperfuzzy Graph (Hyperfuzzy set)[177, 193, 236, 442], HyperNeutrosophic Graph[177], Soft Graph (Soft Set)[34, 222, 294, 314, 462], Hypersoft Graph[178, 182, 382, 383], and Rough Graphs (Rough set) [9, 152, 351–354]. Among these, this paper mainly focuses on Neutrosophic Graphs and Turiyam Neutrosophic Graphs [185–187, 436]. Each graph class has been studied with respect to its specific objectives.

Recently, Neutrosophic Graphs[33, 41, 115, 174, 177, 180, 200, 215, 239, 385, 429, 436] and Neutrosophic Hypergraphs[31, 45, 153, 287] have gained attention within the framework of Neutrosophic Set Theory[61, 443].

”Neutrosophic” refers to a mathematical approach that extends classical and fuzzy logic by incorporating degrees of truth, indeterminacy, and falsity. As generalizations of Fuzzy Graphs[318, 377], Neutrosophic Graphs have garnered significant interest due to their versatility and broad potential for applications, much like Fuzzy Graphs. Various related classes of Neutrosophic Graphs have also been developed and studied, including Bipolar Neutrosophic Graphs[45], Neutrosophic Incidence Graphs[440], Single valued neutrosophic signed graphs[306], Strong Neutrosophic Graphs[240], m-polar neutrosophic graphs[312], Complex neutrosophic hypergraphs[287], Bipolar neutrosophic hypergraphs[45].

Subsequently, the Turiyam Neutrosophic Graph was introduced as an extension of Neutrosophic and Fuzzy Graphs. A Turiyam Neutrosophic Graph is distinguished by each vertex and edge having four attributes: truth, indeterminacy, falsity, and a liberal state, thus expanding the framework of Neutrosophic and Fuzzy Graphs [185–187]. Moreover, a further generalization of these graphs, known as Plithogenic Graphs, has been developed and is also actively studied [241, 412, 449, 451].

Given the extensive and continuously expanding body of literature on fuzzy mathematics, it is inevitable that similar concepts may emerge independently across different journals and periods. Nevertheless, we believe that efforts to unify these concepts are essential and will greatly contribute to advancing the field. Additionally, when conducting research or exploring applications, it is important to consider and investigate a wide range of graph classes to identify the most suitable options. This comprehensive approach is crucial for thorough analysis.

1.3 Our Contribution

Based on the above, research on graphs and graph classes that address uncertainty is of great importance. The study of the Turiyam Neutrosophic Graph class has only just begun and remains far less known compared to Fuzzy Graphs, Intuitionistic Fuzzy Graphs, and Neutrosophic Graphs. This paper explores a new graph class for Intuitionistic Fuzzy Graphs and Turiyam Neutrosophic Graphs: General Intuitionistic Fuzzy Graphs, General Turiyam Neutrosophic Graphs and Turiyam Neutrosophic Hypergraphs. Additionally, we consider Extended Turiyam Neutrosophic Graphs, which are graphs designed to handle five parameters from the perspective of uncertainty. We also examine whether most of the graphs mentioned above can be generalized within the framework of Plithogenic Graphs.

The main result of this paper is presented in the following theorem. In addition to this theorem, we also examined the hierarchy of each graph class. Note that the Turiyam Neutrosophic Set is, in fact, a specific case of the Quadripartitioned Neutrosophic Set, achieved by replacing ”Contradiction” with ”Liberal.” (cf.[417, 434])

Theorem 1.1. *In each graph class, the following relationships hold.*

- *An empty graph and a null graph can be represented as 2-valued graphs and 3-valued graphs.*
- *Every edge-fuzzy graph can be transformed into a 2-valued graph by thresholding the edge membership values.*
- *Every fuzzy graph can be transformed into a 3-valued graph by mapping the fuzzy membership values of vertices and edges to the values {-1, 0, 1}.*
- *Every Intuitionistic Fuzzy Graph can be transformed into a Fuzzy Graph by restricting the non-membership function v_A to 0 for all vertices.*
- *Every Neutrosophic Graph can be transformed into an Intuitionistic Fuzzy Graph by setting the indeterminacy value to zero.*
- *Every Extended Turiyam Neutrosophic Graph is a generalization of the Turiyam Neutrosophic Graph.*
- *A plithogenic Graphs generalize Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, Extended Turiyam Neutrosophic Graphs.*
- *Every general plithogenic Graphs can be transformed into General Turiyam Neutrosophic Graph, General Fuzzy Graph, General Intuitionistic Fuzzy Graph, Four-Valued Fuzzy graph, Ambiguous graph, Picture Fuzzy Graph, Hesitant Fuzzy Graph, Intuitionistic Hesitant Fuzzy Graph, Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Quadripartitioned Neutrosophic graph, Pentapartitioned Neutrosophic graph, Turiyam Neutrosophic Graphs, Extended Turiyam Neutrosophic Graphs, and Spherical Fuzzy Graphs.*

Additionally, when considering the relationships between different graph classes, another goal is to organize the characteristics of this field by proving several other theorems related to each graph class as a byproduct, as well as investigating application examples.

In Section 2, we outline the definitions and characteristics used throughout the paper. This section not only introduces various graph classes, but also provides examples and discussions of other graph classes (some of which are already well-known), along with explanations of their applications. Section 3 defines General Intuitionistic Fuzzy Graphs, Turiyam Neutrosophic Graphs, and Turiyam Neutrosophic Hypergraphs, and explores their relationships with other graph types. In Section 4, we examine Plithogenic Graphs, explaining how Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Extended Turiyam Neutrosophic Graphs can be generalized within the framework of Plithogenic Graphs. Finally, Section 5 briefly discusses the future challenges related to graph parameters in Turiyam Neutrosophic Graphs. Graph parameters, simply put, are measures that indicate how closely a given graph matches the desired structure. As the study of graph classes and structures is essential, so too is the investigation of graph parameters [203, 396].

1.4 The Structure of the Paper

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2. Preliminaries and definitions

In this section, we provide a brief overview of the definitions and notations used throughout this paper. Specifically, we cover fundamental concepts of graphs, hypergraphs, fuzzy graphs, Intuitionistic Fuzzy Graphs, Turiyam Neutrosophic graphs/hypergraphs, Neutrosophic graphs, Plithogenic graphs, and single-valued Neutrosophic graphs. Furthermore, we introduce some key properties associated with each type of graph. Additionally, we highlight relevant applications for each graph class where applicable.

Note that this paper may also address set theory concepts in addition to graph theory. Please refer to relevant surveys or notes on set theory as needed [214, 234, 276].

2.1 Basic Graph Concepts

Here are a few basic graph concepts listed below. For more foundational graph concepts and notations, please refer to [155, 156, 156, 157, 197, 473].

Definition 2.1 (Graph)**.** [157] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Example 2.2. Consider the graph $G = (V, E)$, where the vertex set V is defined as $V = \{A, B, C, D\}$, and the edge set E is defined as $E = \{(A, B), (B, C), (C, D), (A, D)\}$.

Definition 2.3 (Subgraph)**.** [157] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

Example 2.4. A subgraph H of the graph $G = (V, E)$ from the previous example can be formed by selecting the vertex set $V_H = \{A, B, D\}$ and the edge set $E_H = \{(A, B), (A, D)\}$.

Definition 2.5 (Degree)**.** [157] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Example 2.6 (Degree)**.** Consider an undirected graph $G = (V, E)$ where

$$V = \{a, b, c, d\}$$

and

$$E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}\}$$

. The degree of each vertex is as follows:

- $\deg(a) = 3$ because a is connected to b, c, and d.
- $\deg(b) = 2$ as it connects to a and d.
- $\deg(c) = 1$ since it's only connected to a.
- $\deg(d) = 2$ due to connections with a and b.

For a directed graph, let $G' = (V', E')$ with $V' = \{x, y, z\}$ and $E' = \{(x, y), (y, z), (z, x), (y, x)\}$. The in-degrees and out-degrees are:

- $\deg^-(x) = 1, \deg^+(x) = 1$
- $\deg^-(y) = 2, \deg^+(y) = 2$
- $\deg^-(z) = 1, \deg^+(z) = 1$

Definition 2.7 (Path)**.** (cf.[483]) A path in a graph $G = (V, E)$ is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $i = 1, 2, \ldots, k-1$. A path is represented as:

$$P = (v_1, v_2, \ldots, v_k),$$

where no vertex is repeated. The length of a path is the number of edges it contains, i.e., $k - 1$.

Example 2.8 (Path)**.** Consider an undirected graph $G = (V, E)$ with $V = \{p, q, r, s\}$ and $E = \{\{p, q\}, \{q, r\}, \{r, s\}\}$. A path from vertex p to vertex s is represented as:

$$P = (p, q, r, s)$$

The length of this path is 3, corresponding to the 3 edges traversed: $\{p, q\}, \{q, r\}, \{r, s\}$.

Definition 2.9 (Cycle)**.** (cf.[483]) A cycle is a closed path, meaning it is a path where $v_1 = v_k$, forming a loop with no other repeated vertices. Formally, a cycle is a sequence:

$$C = (v_1, v_2, \ldots, v_k, v_1),$$

where $v_1 = v_k$, and $\{v_i, v_{i+1}\} \in E$ for $i = 1, 2, \ldots, k-1$.

Example 2.10 (Cycle)**.** In the same graph $G = (V, E)$ as above, add an edge $\{s, p\}$ to form a new set of edges $E = \{\{p, q\}, \{q, r\}, \{r, s\}, \{s, p\}\}$. Now there exists a cycle:

$$C = (p, q, r, s, p)$$

This sequence starts and ends at p, creating a closed loop, and no other vertex is repeated within the cycle.

Definition 2.11 (Acyclicity)**.** A graph $G = (V, E)$ is called *acyclic* if it contains no cycles. That is, for any sequence of distinct vertices $v_1, v_2, \ldots, v_k$ with $k \geq 3$, there does not exist a set of edges

$$\{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)\}$$

that form a cycle.

Definition 2.12 (Tree)**.** (cf.[483]) A tree is a connected, acyclic graph. In other words, a tree is a graph where there is exactly one path between any two vertices, and no cycles exist.

Example 2.13. Here is an example of a Tree based on the provided definitions:

$$T = (V, E), \quad V = \{v_1, v_2, v_3, v_4, v_5\}, \quad E = \{(v_1, v_2), (v_1, v_3), (v_3, v_4), (v_3, v_5)\}$$

This graph T has the following properties:

- The set of vertices $V = \{v_1, v_2, v_3, v_4, v_5\}$ represents five distinct vertices.
- The set of edges $E = \{(v_1, v_2), (v_1, v_3), (v_3, v_4), (v_3, v_5)\}$ connects these vertices.
- T is connected, meaning there is a path between any pair of vertices.
- T is acyclic, meaning it contains no cycles.

Thus, T satisfies the definition of a *Tree*, as it is connected and acyclic. For example, there is exactly one path from v_1 to v_5 through v_3, and no cycles are formed.

2.1.1 Useful graph class

We introduce some representative classical graph classes. These graph classes have also been studied in the contexts of fuzzy, neutrosophic, and plithogenic settings.

Definition 2.14. (cf.[128, 275, 481]) The **line graph** $L(G)$ of a given graph $G = (V, E)$ is a graph constructed as follows:

- Each vertex in $L(G)$ represents an edge in the original graph G.
- Two vertices in $L(G)$ are connected by an edge if and only if their corresponding edges in G share a common vertex.

Definition 2.15. (cf.[143, 376, 460]) A graph $G = (V, E)$ is called a **planar graph** if it can be drawn on a plane in such a way that no two edges intersect except at their endpoints. In other words, there exists a way to represent the graph on a two-dimensional plane such that the edges only meet at shared vertices, without any of the edges crossing each other.

Definition 2.16. (cf.[198,329]) An **outer-planar graph** $G = (V, E)$ is an undirected graph that can be embedded in the plane such that all vertices lie on the outer face of the embedding, meaning no vertex is entirely enclosed by edges. In other words, it is possible to draw the graph on a plane without any edge crossings, with all vertices positioned on the unbounded (outer) face of the graph.

1. **Edge Crossing Condition**: The graph G must be drawn in such a way that no two edges intersect except at their endpoints.
2. **Outer Face Requirement**: All vertices of G must lie on the boundary of a single, unbounded face of the graph.

2.1.2 Empty Graph and Null graph

The definitions of an Empty Graph and a Null Graph are provided below.

Definition 2.17 (Empty Graph and Null graph)**.** (cf.[208, 400]) An *empty graph* is a graph that contains a set of vertices V but has no edges. Formally, an empty graph $G = (V, E)$ is defined as a graph where $E = \emptyset$, meaning the edge set is empty. This graph has no connections between its vertices. A *null graph* is a graph with no vertices and no edges. Formally, a null graph is a graph $G = (V, E)$ where both the vertex set $V = \emptyset$ and the edge set $E = \emptyset$. It is the simplest possible graph structure, containing no elements.

I will describe the operations of graph vertex addition and graph edge addition. These are commonly used operations in graph algorithms (ex.[104, 145, 211]).

Definition 2.18 (Graph Vertex Addition)**.** Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. **Graph vertex addition** is the operation of adding a new vertex v_{new} to the graph G, resulting in a new graph $G' = (V', E')$, where:

$$V' = V \cup \{v_{\text{new}}\}, \quad E' = E.$$

No new edges are introduced in this operation. This operation is commonly used in graph algorithms to expand the graph by introducing additional vertices.

Definition 2.19 (Graph Edge Addition)**.** Let $G = (V, E)$ be a graph. **Graph edge addition** is the operation of adding a new edge $e_{\text{new}} = (u, v)$, where $u, v \in V$, to the edge set E of the graph G, resulting in a new graph $G' = (V', E')$, where:

$$V' = V, \quad E' = E \cup \{e_{\text{new}}\}.$$

This operation is commonly used in graph algorithms to create connections between existing vertices.

Proposition 2.20. *If a vertex addition is performed on a null graph, the result is an empty graph.*

Proof. A *null graph* is defined as a graph with no vertices and no edges, formally represented as $G = (V, E)$, where $V = \emptyset$ and $E = \emptyset$.

Let us perform a *vertex addition* on this null graph by adding a single vertex v_{new}. After the addition of v_{new}, the updated vertex set V' becomes $V' = \{v_{\text{new}}\}$. The edge set remains unchanged because no edges are introduced during vertex addition, so $E' = \emptyset$.

Thus, the resulting graph $G' = (V', E')$ is a graph with one vertex and no edges. By definition, a graph with one or more vertices and no edges is called an *empty graph*. Therefore, the result of adding a vertex to a null graph is an empty graph. □

Proposition 2.21. *If an edge addition is performed on an empty graph, the result is a graph.*

Proof. An *empty graph* is defined as a graph with a set of vertices V, but no edges, i.e., $E = \emptyset$. Formally, an empty graph is represented as $G = (V, E)$, where $|V| \geq 1$ and $E = \emptyset$.

Let us perform an *edge addition* on this empty graph by adding an edge $e_{\text{new}} = (u, v)$, where $u, v \in V$. After the addition of the edge e_{new}, the updated edge set E' becomes $E' = \{e_{\text{new}}\}$, while the vertex set V' remains the same as $V' = V$.

The resulting graph $G' = (V', E')$ has the same set of vertices V, and now it has at least one edge $E' \neq \emptyset$. By definition, any graph with vertices and at least one edge is simply called a *graph*. Therefore, the result of adding an edge to an empty graph is a graph. □

2.1.3 2-Valued Graph and 3-Valued Graph

In this subsubsection, we provide an explanation of 2-Valued Graphs and 3-Valued Graphs. In graph theory, vertices and edges are often represented as either existing (denoted by 1) or not existing (denoted by 0). It is worth noting that the concept of a 2-Valued Graph is highly similar to the definition of a Crisp Graph[97, 307, 335, 358] commonly used in the field of Fuzzy Graphs.

When expressed as a 2-valued function, it takes the following form:

Definition 2.22. In a graph $G = (V, E)$, where V is the set of vertices and E is the set of edges, we can define a 2-valued function for both vertices and edges as follows:

- The function $f_V : V \to \{0, 1\}$ represents the state of each vertex $v \in V$:

$$f_V(v) = \begin{cases} 1 & \text{if vertex } v \text{ exists,} \\ 0 & \text{if vertex } v \text{ does not exist.} \end{cases}$$

- Similarly, the function $f_E : E \to \{0, 1\}$ represents the state of each edge $e \in E$:

$$f_E(e) = \begin{cases} 1 & \text{if edge } e \text{ exists,} \\ 0 & \text{if edge } e \text{ does not exist.} \end{cases}$$

To put it bluntly, the idea of extending the functions assigned to vertices and edges—such as by using membership functions or other types of functions—is a fundamental approach to handling real-world concepts and uncertainty. This makes the concept easier to understand.

Proposition 2.23. *An empty graph and a null graph can be represented as 2-valued graphs.*

Proof. Empty graph: Consider a graph $G = (V, E)$ where the vertex set V exists, but the edge set $E = \emptyset$. This is an empty graph. For each vertex $v \in V$, we define the following 2-valued function f_V:

$$f_V(v) = \begin{cases} 1 & \text{if vertex } v \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

For the edge set E, we define a 2-valued function $f_E(e) = 0$ (since no edges exist in the empty graph).

Null graph: In a null graph $G = (V = \emptyset, E = \emptyset)$, there are no vertices or edges. As such, the 2-valued functions f_V and f_E do not apply, effectively representing an empty graph structure.

Thus, both the empty graph and the null graph can be represented using 2-valued functions. □

Here is a rough example, but let's consider a 3-valued function (cf.[213, 282, 299]). We can define a 3-valued function for both vertices and edges in a graph $G = (V, E)$ as follows:

Definition 2.24. In a graph $G = (V, E)$, where V is the set of vertices and E is the set of edges, a 3-valued function can be defined for both vertices and edges as follows:

- **For vertices**: Let $f_V : V \to \{-1, 0, 1\}$ represent the state of each vertex $v \in V$. The function assigns values as follows:

$$f_V(v) = \begin{cases} 1 & \text{if vertex } v \text{ is in a positive state (exists),} \\ 0 & \text{if vertex } v \text{ is neutral,} \\ -1 & \text{if vertex } v \text{ is in a negative state (non-existent or unfavorable).} \end{cases}$$

- **For edges**: Similarly, define $f_E : E \to \{-1, 0, 1\}$ to represent the state of each edge $e \in E$:

$$f_E(e) = \begin{cases} 1 & \text{if edge } e \text{ is in a positive state (exists),} \\ 0 & \text{if edge } e \text{ is neutral,} \\ -1 & \text{if edge } e \text{ is in a negative state (non-existent or unfavorable).} \end{cases}$$

Theorem 2.25. *A 3-valued graph is a generalization of a 2-valued graph.*

Proof. This is evident. □

In this paper, we consider graphs with uncertainty incorporated into them.

2.2 Hypergraph Concepts

A hypergraph is a generalization of a graph where edges, called hyperedges, can connect any number of vertices, not just two. This structure is useful for modeling complex relationships in various fields like computer science and biology[170, 195, 196, 355]. The definition is provided below.

Definition 2.26. [106] A hypergraph is a pair $H = (V(H), E(H))$, consisting of a nonempty set $V(H)$ of vertices and a set $E(H)$ of subsets of $V(H)$, called the hyperedges of H. In this paper, we consider only finite hypergraphs.

Example 2.27. Let H be a hypergraph with vertex set $V(H) = \{A, B, C, D, E\}$ and hyperedge set $E(H) = \{e_1, e_2, e_3\}$, where:

$$e_1 = \{A, D\}, \quad e_2 = \{D, E\}, \quad e_3 = \{A, B, C\}.$$

Thus, H is represented by the pair $H(V, E) = (\{A, B, C, D, E\}, \{\{A, D\}, \{D, E\}, \{A, B, C\}\})$.

Theorem 2.28. *Any classical (binary) graph $G = (V, E)$, where V is the set of vertices and E is the set of edges, can be represented as a hypergraph $H = (V(H), E(H))$, where $V(H) = V$ and each edge $e \in E$ corresponds to a hyperedge in $E(H)$ consisting of two vertices.*

Proof. Let $G = (V, E)$ be a classical graph, where:

- $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices, and
- $E = \{e_1, e_2, \ldots, e_m\}$ is the set of edges, where each edge $e_i = (u_i, v_i)$ connects two vertices $u_i, v_i \in V$.

We define a corresponding hypergraph $H = (V(H), E(H))$, where:

- $V(H) = V$, meaning the vertex set of the hypergraph is the same as the vertex set of the graph.
- $E(H) = \{e_1, e_2, \ldots, e_m\}$, where each hyperedge $e_i \in E(H)$ is a 2-element subset of $V(H)$, corresponding to the edge $e_i = (u_i, v_i)$ in the graph.

For each edge $e_i = (u_i, v_i) \in E$ in the graph, we define a hyperedge $e_i' = \{u_i, v_i\} \in E(H)$ in the hypergraph. Thus, the hypergraph H is constructed such that each edge in the original graph is represented by a hyperedge containing exactly two vertices.

Since each edge in a graph connects exactly two vertices, and each hyperedge in the constructed hypergraph contains exactly two vertices, this hypergraph representation faithfully captures the structure of the original graph.

The classical graph $G = (V, E)$ is equivalent to a hypergraph $H = (V(H), E(H))$ where every edge in the graph corresponds to a 2-element hyperedge in the hypergraph. Therefore, any graph can be represented as a hypergraph where each hyperedge contains exactly two vertices. □

Definition 2.29. [106] For a hypergraph H and a subset $X \subseteq V(H)$, the subhypergraph induced by X is defined as $H[X] = (X, \{e \cap X \mid e \in E(H)\})$. We denote the hypergraph obtained by removing X from H as $H \setminus X := H[V(H) \setminus X]$.

This paper does not delve into detailed discussions, but Supergraphs [98] and Superhypergraph [73, 175, 181, 192, 229, 323, 360, 428, 429, 435, 439] are well-known graph classes that are closely related to both graphs and hypergraphs. The following diagram illustrates their relationship with these graph classes. The same applies to directed graphs as well.

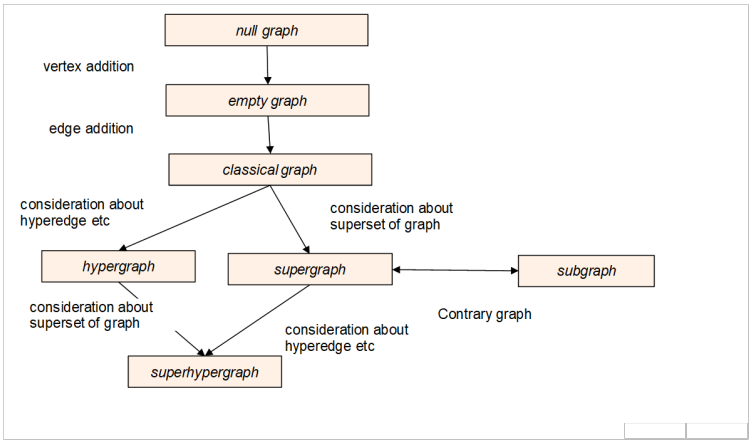

Fig. 1. Graph Hierarchy for Hypergraph

2.3 Fuzzy Graph

A Fuzzy Graph captures relationships involving uncertainty by assigning membership degrees to both vertices and edges, enabling flexible and nuanced analysis. It can also be viewed as a graphical representation of a fuzzy set (cf. [1, 161, 281, 357, 477]). Due to its significance, Fuzzy Graphs have been the subject of extensive research [32, 37, 48, 51, 54, 56, 93, 95, 237, 283, 316, 322, 447, 470].

The Fuzzy framework encompasses a wide range of mathematical and logical structures that have been extensively studied and developed in recent years. The primary concepts include:

1. Fuzzy Set [261, 293, 365, 477, 484]
2. Fuzzy Topological Spaces [285, 444]
3. Fuzzy Logics [90, 204]
4. Fuzzy Algebraic Structures [243]
5. Fuzzy Environment [89, 330]
6. Fuzzy Geometry [378]
7. Fuzzy Statistics [243]
8. Fuzzy Physics [86]
9. Fuzzy Control [160, 342, 356]
10. Fuzzy system [124, 132]

2.3.1 Basic concepts for fuzzy graph

In this subsubsection, we explain the basic concepts of fuzzy graphs. A Fuzzy Graph is often discussed in relation to a crisp graph [97, 307, 335, 358]. First, we provide the definition of a crisp graph below.

Definition 2.30. A crisp graph $G = (V, E)$ consists of V, a non-empty set of vertices (or nodes), and E, a set of edges. Each edge is associated with either one or two vertices, which are referred to as its endpoints. An edge is said to connect its endpoints.

To put it simply, a fuzzy graph can be seen as a crisp graph with the concept of fuzzy relation applied to it. The definition of a fuzzy relation is presented below.

Definition 2.31. Let S be a non-empty set. A **fuzzy relation** on S is defined as a fuzzy subset $\mu : S \times S \to [0, 1]$, which assigns a membership value to each pair of elements in S. The membership value $\mu(x, y)$ represents the degree of relationship between elements x and y in S.

Let $\sigma : S \to [0, 1]$ be a fuzzy set on S, where $\sigma(x)$ represents the degree of membership of element x in the fuzzy set. The fuzzy relation μ is said to be a relation on σ if, for all $x, y \in S$, the following condition holds:

$$\mu(x, y) \leq \sigma(x) \wedge \sigma(y),$$

where $\wedge$ denotes the minimum operation, i.e.,

$$\sigma(x) \wedge \sigma(y) = \min\{\sigma(x), \sigma(y)\}.$$

In this context, $\mu(x, y)$ indicates the strength of the relationship between x and y, and the condition ensures that the relationship between x and y does not exceed the membership values of x and y in the fuzzy set σ.

The formal definition of a Fuzzy Graph is as follows [317, 377].

Definition 2.32. [377] A fuzzy graph $G = (\sigma, \mu)$ with V as the underlying set is defined as follows:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of vertices, where $\sigma(x)$ represents the membership degree of vertex $x \in V$.
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation on σ, such that $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ denotes the minimum operation.

The underlying crisp graph of G is denoted by $G^* = (\sigma^*, \mu^*)$, where:

- $\sigma^* = \sup p(\sigma) = \{x \in V : \sigma(x) > 0\}$
- $\mu^* = \sup p(\mu) = \{(x, y) \in V \times V : \mu(x, y) > 0\}$

Definition 2.33. [377] A fuzzy subgraph $H = (\sigma', \mu')$ of G is defined as follows:

- There exists $X \subseteq V$ such that $\sigma' : X \to [0, 1]$ is a fuzzy subset.
- $\mu' : X \times X \to [0, 1]$ is a fuzzy relation on σ', satisfying $\mu'(x, y) \leq \sigma'(x) \wedge \sigma'(y)$ for all $x, y \in X$.

Example 2.34. (cf.[116]) Consider a fuzzy graph $G = (\sigma, \mu)$ with four vertices $V = \{v_1, v_2, v_3, v_4\}$, as depicted in the diagram.

The membership degrees of the vertices are as follows:

$$\sigma(v_1) = 0.1, \quad \sigma(v_2) = 0.3, \quad \sigma(v_3) = 0.2, \quad \sigma(v_4) = 0.4$$

The fuzzy relation on the edges is defined by the values of μ, where $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$. The fuzzy membership degrees of the edges are as follows:

$$\mu(v_1, v_2) = 0.1, \quad \mu(v_2, v_3) = 0.1, \quad \mu(v_3, v_4) = 0.1$$

$$\mu(v_4, v_1) = 0.1, \quad \mu(v_2, v_4) = 0.3$$

In this case, the fuzzy graph G has the following properties:

- Vertices v_1, v_2, v_3, v_4 are connected by edges with varying membership degrees.
- The fuzzy relations ensure that $\mu(x, y)$ for any edge (x, y) does not exceed the minimum membership of the corresponding vertices.

The definitions of "Complete" and "Strong," which are frequently used properties of fuzzy graphs, are provided below.

Definition 2.35. [189] A fuzzy graph $G = (\sigma, \mu)$ is called *complete* if for all $u, v \in V$, the following condition holds:

$$\mu(u, v) = \sigma(u) \wedge \sigma(v),$$

where $\wedge$ denotes the minimum operation.

Definition 2.36. [189] A fuzzy graph $G = (\sigma, \mu)$ is called *strong* if for all $u, v \in E$, the same condition holds:

$$\mu(u, v) = \sigma(u) \wedge \sigma(v).$$

Theorem 2.37. *Every complete fuzzy graph is a strong fuzzy graph, but not every strong fuzzy graph is a complete fuzzy graph.*

Proof. By definition, a complete fuzzy graph $G = (\sigma, \mu)$ satisfies:

$$\mu(u, v) = \sigma(u) \wedge \sigma(v) \quad \forall u, v \in V,$$

where $\wedge$ is the minimum operation. This condition holds for all pairs of vertices, ensuring that every complete fuzzy graph is a strong fuzzy graph, as a strong fuzzy graph also satisfies:

$$\mu(u, v) = \sigma(u) \wedge \sigma(v) \quad \forall u, v \in E.$$

Thus, every complete fuzzy graph is strong.

A strong fuzzy graph only requires the condition $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for edges $(u, v) \in E$, but does not require edges between every pair of vertices. In contrast, a complete fuzzy graph requires edges between all pairs of vertices. Therefore, not every strong fuzzy graph is complete. □

2.3.2 Other graph classes related to fuzzy graph

Next, we will examine the relationships with other classes of graphs.

Proposition 2.38. *Every Fuzzy Graph can be transformed into an null graph by restricting the membership values of vertices and edges to* 0.

Proof. Let $G = (V, E, \sigma, \mu)$ be a fuzzy graph, where $\sigma : V \to [0, 1]$ represents the membership function of the vertices, and $\mu : V \times V \to [0, 1]$ represents the membership function of the edges.

To transform G into a null graph (a graph with no vertices or edges), we need to set the membership values of all vertices and edges to 0.

Formally, for each vertex $v \in V$ and each edge $(u, v) \in E$, we redefine:

$$\sigma(v) = 0 \quad \text{and} \quad \mu(u, v) = 0.$$

This operation results in no vertices or edges having a non-zero membership value, thereby reducing the graph to a null graph. □

Proposition 2.39. *Every Fuzzy Graph can be transformed into an empty graph by restricting the membership values of edges to* 0.

Proof. Let $G = (V, E, \mu)$ be a fuzzy graph, where μ is the membership function of the edges. By definition, $\mu : V \times V \to [0, 1]$ assigns a membership value to each edge.

To transform G into an empty graph, we simply set the membership values of all edges to 0. Formally, redefine $\mu(u, v) = 0$ for all $(u, v) \in E$.

Since an empty graph has no edges, this operation effectively removes all connections, making the resulting graph an empty graph with only isolated vertices. □

Proposition 2.40. *Every Fuzzy Graph can be transformed into a Crisp Graph by restricting the membership values of vertices and edges to* 0 *or* 1.

Proof. Let $G_F = (V, E, \mu_V, \mu_E)$ be a Fuzzy Graph, where:

- $\mu_V : V \to [0, 1]$ assigns a membership value to each vertex $v \in V$,
- $\mu_E : E \to [0, 1]$ assigns a membership value to each edge $(u, v) \in E$.

We want to transform G_F into a Crisp Graph $G_C = (V_C, E_C)$, where:

- $V_C \subseteq V$ is the set of vertices in the Crisp Graph,
- $E_C \subseteq V_C \times V_C$ is the set of edges in the Crisp Graph.

The transformation is performed as follows:

1. Vertices:
$$V_C = \{v \in V : \mu_V(v) > 0\}.$$
If a vertex $v \in V$ has a positive membership value $\mu_V(v) > 0$, then v is included in the vertex set V_C of the Crisp Graph. If $\mu_V(v) = 0$, then v is excluded.

2. Edges:
$$E_C = \{(u, v) \in E : \mu_E(u, v) > 0\}.$$
Similarly, if an edge $(u, v) \in E$ has a positive membership value $\mu_E(u, v) > 0$, then (u, v) is included in the edge set E_C of the Crisp Graph. If $\mu_E(u, v) = 0$, then the edge is excluded.

Since the membership values in G_C are restricted to 0 or 1, this transformation converts the Fuzzy Graph into a Crisp Graph. □

Proposition 2.41. *Every fuzzy graph can be transformed into a 3-valued graph by mapping the fuzzy membership values of vertices and edges to the values {-1, 0, 1}.*

Proof. Let $G_F = (V, E, \sigma, \mu)$ be a fuzzy graph, where:

- $\sigma : V \to [0, 1]$ assigns membership values to vertices, and
- $\mu : E \to [0, 1]$ assigns membership values to edges.

To transform G_F into a 3-valued graph $G_T = (V_T, E_T, f_V, f_E)$, we define the following mappings for vertices and edges:

Define a function $f_V : V \to \{-1, 0, 1\}$ for each vertex $v \in V$ as follows:

$$f_V(v) = \begin{cases} 1 & \text{if } \sigma(v) > 0.5 \text{ (positive state)}, \\ 0 & \text{if } \sigma(v) = 0.5 \text{ (neutral state)}, \\ -1 & \text{if } \sigma(v) < 0.5 \text{ (negative state)}. \end{cases}$$

For edges: Similarly, define a function $f_E : E \to \{-1, 0, 1\}$ for each edge $e \in E$ as follows:

$$f_E(e) = \begin{cases} 1 & \text{if } \mu(e) > 0.5 \text{ (positive state)}, \\ 0 & \text{if } \mu(e) = 0.5 \text{ (neutral state)}, \\ -1 & \text{if } \mu(e) < 0.5 \text{ (negative state)}. \end{cases}$$

By applying these mappings, the fuzzy membership values $\sigma(v)$ and $\mu(e)$, which lie in the interval $[0, 1]$, are transformed into discrete values $\{-1, 0, 1\}$ corresponding to negative, neutral, and positive states. This preserves the uncertainty modeled by the fuzzy graph while allowing for a simpler 3-valued representation.

Thus, every fuzzy graph can be transformed into a 3-valued graph by appropriately mapping its membership values. □

2.3.3 Edge-fuzzy Graph

The above definition of a fuzzy graph considers membership values for both vertices and edges. However, there are also definitions that consider membership values solely for vertices or solely for edges. These are known as vertex fuzzy graphs and edge fuzzy graphs, respectively (cf.[159]).

Definition 2.42. An **Edge-fuzzy Graph** $G = (V, E, \mu)$ is a graph where:

- V is the set of vertices.
- $E \subseteq V \times V$ is the set of edges.
- $\mu : E \to \mathbb{R}$ is a function that assigns a real value to each edge. The value $\mu(e) \in \mathbb{R}$ represents the weight, cost, or strength associated with the edge e.

Proposition 2.43. *Let $G = (\sigma, \mu)$ be a fuzzy graph. If the vertex membership function σ is fixed to a constant $c \in [0, 1]$, then the graph can be transformed into an edge-fuzzy graph $G' = (V, \mu')$.*

Proof. In a fuzzy graph $G = (\sigma, \mu)$, the membership value of an edge (u, v) is constrained by the vertex membership values as follows:
$$\mu(u, v) \leq \sigma(u) \wedge \sigma(v).$$
If we fix the membership values of all vertices to a constant $c \in [0, 1]$, then for all vertices $u, v \in V$, we have $\sigma(u) = \sigma(v) = c$. Therefore, the constraint simplifies to:
$$\mu(u, v) \leq c.$$

In the resulting edge-fuzzy graph $G' = (V, \mu')$, we define the edge membership function μ' as:

$$\mu'(u, v) = \frac{\mu(u, v)}{c}.$$

This definition ensures that $\mu'(u, v) \in [0, 1]$, which satisfies the requirements of an edge-fuzzy graph.

Hence, by fixing the vertex membership function σ to a constant c, the fuzzy graph can be transformed into an edge-fuzzy graph. □

Theorem 2.44. *Every edge-fuzzy graph can be transformed into a 2-valued graph by thresholding the edge membership values.*

Proof. Let $G = (V, E, \mu)$ be an edge-fuzzy graph, where $\mu(e) \in [0, 1]$ represents the membership value of edge $e \in E$. To transform this edge-fuzzy graph into a 2-valued graph, we can apply a threshold to the edge membership values, setting a threshold $\tau \in [0, 1]$ to decide whether an edge exists or not in the resulting 2-valued graph.

Define the edge set E_2 of the 2-valued graph G_2 as:

$$E_2 = \{e \in E \mid \mu(e) \geq \tau\}.$$

The function $f_E : E_2 \to \{0, 1\}$ is then defined as:

$$f_E(e) = \begin{cases} 1 & \text{if } \mu(e) \geq \tau, \\ 0 & \text{if } \mu(e) < \tau. \end{cases}$$

This mapping ensures that any edge $e \in E$ with a membership value $\mu(e) \geq \tau$ is included in the 2-valued graph as an existing edge, while edges with $\mu(e) < \tau$ are considered non-existent (i.e., their value is 0).

Thus, the edge-fuzzy graph G has been successfully transformed into a 2-valued graph G_2 based on the threshold τ. □

Based on the previous discussion, it is crucial to determine where to apply the fuzzy concept within the graph and which parts to keep fixed. In the literature [97], the graph types were defined as follows.

Definition 2.45 (Fuzzy Graph Type)**.** [97] A fuzzy graph GF is a graph that satisfies one of the following types of fuzziness (referred to as GF of the i-th type) or any combination thereof:

(i) $GF_1 = \{G_1, G_2, G_3, \ldots, G_F\}$ where fuzziness exists in each graph G_i.

(ii) $GF_2 = \{V, E_F\}$ where the edge set E_F is fuzzy.

(iii) $GF_3 = \{V, E(t_F, h_F)\}$ where both the vertex set V and edge set E are crisp, but the edges have fuzzy heads $h(e_i)$ and fuzzy tails $t(e_i)$.

(iv) $GF_4 = \{V_F, E\}$ where the vertex set V_F is fuzzy.

(v) $GF_5 = \{V, E(w_F)\}$ where both the vertex set V and edge set E are crisp, but the edges have fuzzy weights w_F.

2.3.4 N-graph and general fuzzy graph

In this subsubsection, we explain N-graphs and general fuzzy graphs. Various variants of fuzzy graphs have been defined due to their importance. The N-graph and general fuzzy graph are well-known concepts related to fuzzy graphs. Their definitions are presented below [14, 336].

Definition 2.46. [14] By an N-graph G of a graph, we mean a pair $G = (V, E)$, where V is an N-function on the vertex set V and E is an N-relation on the edge set E, such that for all $u, v \in V$, the relation $(u, v) \leq \max\{N(u), N(v)\}$.

Theorem 2.47. *(cf.[14]) Every N-graph can be transformed into a 2-valued graph by setting the N-function values to* 1 *for all vertices and edges where the membership is non-zero.*

Proof. Let $G = (V, E)$ be an N-graph with the function $N : V \to [0, 1]$ on vertices and the relation $N : E \to [0, 1]$ on edges. To transform G into a 2-valued graph, we need to modify the function values such that the new graph $G_2 = (V, E_2)$ only contains edges and vertices with function values 1 or 0.

Define the vertex function $f_V : V \to \{0, 1\}$ as follows:

$$f_V(v) = \begin{cases} 1 & \text{if } N(v) > 0, \\ 0 & \text{if } N(v) = 0. \end{cases}$$

This ensures that every vertex with a non-zero membership in the N-graph is mapped to 1 in the 2-valued graph.

For the edges, define the edge function $f_E : E \to \{0, 1\}$ as:

$$f_E(u, v) = \begin{cases} 1 & \text{if } N(u, v) > 0, \\ 0 & \text{if } N(u, v) = 0. \end{cases}$$

This transformation ensures that any edge with a non-zero relation in the N-graph is mapped to 1, and edges with zero relation are mapped to 0.

The relation $N(u, v) \leq \max\{N(u), N(v)\}$ still holds after transformation, as the maximum value of $N(u)$ and $N(v)$ is now either 1 or 0. Hence, the transformed edge function $f_E(u, v)$ is consistent with the original relation in the N-graph.

Thus, by setting all non-zero function values to 1, the N-graph has been successfully transformed into a 2-valued graph. □

Definition 2.48. [336] A *general fuzzy graph* with V as the underlying set is defined as a pair of functions $G = (\sigma, \mu)$, where:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of V,
- $\mu : V \times V \to [0, 1]$ is a fuzzy subset of $V \times V$.

This definition generalizes the traditional fuzzy graph by allowing the membership values of edges to be independent of the membership values of their incident vertices.

Definition 2.49. [336] A *general weak fuzzy graph* with V as the underlying set is defined as a pair of functions $G = (\sigma, \mu)$, where:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of V,
- $\mu : V \times V \to [0, 1]$ is a fuzzy subset of $V \times V$,

such that for all $(u, v) \in E$, the membership degree of the edge $\mu(u, v)$ satisfies:

$$\mu(u, v) \neq \sigma(u) \wedge \sigma(v),$$

where $\wedge$ denotes the minimum operation.

Proposition 2.50. *(cf.[336]) Every Weak General fuzzy Graph is a General fuzzy Graph.*

Proof. By the definition of a Weak General fuzzy Graph, for all vertices $u, v \in V$ and edges $(u, v) \in E$, the following condition holds:

$$\mu(u, v) \neq \sigma(u) \wedge \sigma(v),$$

where $\wedge$ denotes the minimum operation.

Now, consider the definition of a *General fuzzy Graph*. A General fuzzy Graph $G = (\sigma, \mu)$ consists of the same functions:

- $\sigma : V \to [0, 1]$,
- $\mu : V \times V \to [0, 1]$,

but without the restriction that $\mu(u, v)$ must equal $\sigma(u) \wedge \sigma(v)$. In other words, $\mu(u, v)$ can take any value independent of the membership degrees of the vertices.

Since the condition $\mu(u, v) \neq \sigma(u) \wedge \sigma(v)$ is less restrictive than requiring $\mu(u, v) = \sigma(u) \wedge \sigma(v)$, any Weak General fuzzy Graph satisfies the conditions of a General fuzzy Graph.

Thus, every Weak General fuzzy Graph is a General fuzzy Graph. □

Proposition 2.51. *(cf.[336]) Every fuzzy Graph is a Weak General fuzzy Graph.*

Proof. For a fuzzy Graph, the membership degree of each edge $\mu(u, v)$ is required to satisfy:

$$\mu(u, v) = \sigma(u) \wedge \sigma(v),$$

where $\wedge$ is the minimum operation.

Now, consider the definition of a *Weak General fuzzy Graph*. In a Weak General fuzzy Graph, there is no requirement for $\mu(u, v)$ to be exactly $\sigma(u) \wedge \sigma(v)$, but instead, it only requires that $\mu(u, v) \neq \sigma(u) \wedge \sigma(v)$.

Since every fuzzy Graph satisfies $\mu(u, v) = \sigma(u) \wedge \sigma(v)$, which is a specific case of $\mu(u, v) \neq \sigma(u) \wedge \sigma(v)$ (in the sense that it does not violate the condition), every fuzzy Graph also satisfies the conditions of a Weak General fuzzy Graph.

Thus, every fuzzy Graph is a Weak General fuzzy Graph. □

2.3.5 Related graph class for fuzzy graph

In this subsubsection, we explain Related graph classes for fuzzy graph. In the field of Fuzzy Graphs, numerous graph classes have also been proposed. Here, we briefly introduce the related graph classes for fuzzy graphs.

Notation 2.52. In this paper, we define the term ”Related graph class” as a graph class that either extends or restricts a corresponding graph class in some way.

Theorem 2.53. *The following are examples of related graph classes, including but not limited to:*

- *Bipolar Fuzzy Graphs [15]*
- *Fuzzy Planar Graphs [392]*
- *Irregular Bipolar Fuzzy Graphs [391]*
- *Regular Bipolar Fuzzy Graphs [23]*
- *Picture Fuzzy Tolerance Graphs [135]*
- *Complex Hesitant Fuzzy Graphs [6]*
- *Strong Intuitionistic Fuzzy Graphs [21]*
- *Product Fuzzy Graphs [53]*
- *Partially Total Fuzzy Graphs [8]*
- *Fuzzy Influence Graphs [304]*
- *Picture Fuzzy Directed Hypergraphs [255]*
- *Radio Fuzzy Graphs [290]*
- *Line Regular Fuzzy Semigraphs [74]*
- *Fuzzy Incidence Graphs [158]*
- *Balanced Picture Fuzzy Graphs [66]*
- *Oscillating Polar Fuzzy Graphs [65]*
- *Cayley Fuzzy Graphs [457]*
- *Rough Fuzzy Digraphs [9]*
- *T-Spherical Fuzzy Graphs [199]*
- *Mixed Fuzzy Graphs [387]*
- *Einstein Fuzzy Graphs [228]*
- *Edge-Regular Fuzzy Graphs [55, 118, 339]*
- *Robust Fuzzy Graphs [448]*
- *Anti-Product Fuzzy Graphs [49]*
- *Valued Fuzzy Superhypergraphs [433]*
- *Inverse Fuzzy Graphs [100]*
- *Inverse Eccentric Fuzzy Graphs [305]*
- *Cubic Pythagorean Fuzzy Graphs [321]*

- *Complete Fuzzy Graphs [50]*
- *Mixed Picture Fuzzy Graphs [327]*
- *Extended Total Fuzzy Graphs [7]*
- *Pseudo Regular Fuzzy Graphs [292]*
- *Best Fuzzy Graphs [452]*
- *Intuitionistic Felicitous Fuzzy Graphs [94]*
- *Middle Fuzzy Graphs [309, 450]*
- *Bipolar Fuzzy P-Competition Graphs [337]*
- *Fuzzy Intersection Graphs [372]*
- *Fuzzy Semigraphs [380]*
- *Intuitionistic Fuzzy Soft Expert Graphs [464]*
- *m-Polar Fuzzy Graphs [17]*
- *Balanced Interval-Valued Fuzzy Graphs [370]*
- *Double Layered Fuzzy graph[260]*
- *Triple Layered Fuzzy Graph[379]*
- *Fuzzy Outerplanar Graphs [176, 231]*
- *Inverse fuzzy multigraphs[99]*
- *Bipolar inverse fuzzy graphs[253]*
- *Fuzzy zero divisor graphs [269, 270]*

Considering these fuzzy graph classes enables the identification of shared properties, which can lead to the development of efficient algorithms, deeper analysis, and practical applications across various fields.

Proof. Refer to each reference as needed. □

2.3.6 Application for fuzzy graph

In this subsubsection, we explain Applications for fuzzy graph. Fuzzy graphs have a wide range of applications. Here, we introduce a few practical examples.

- Neural Network: A neural network is a computational model inspired by the human brain, using interconnected nodes (neurons) to process data and learn patterns(cf.[5,202]). Various studies have been conducted on the application of fuzzy graphs to neural networks [267, 283, 472, 478].
- Decision-Making: Decision-making using graphs involves representing problems visually with nodes and edges, allowing for structured analysis of options, dependencies, and outcomes to reach optimal solutions (cf.[259,278,445]). Fuzzy graphs and decision-making are highly compatible, and numerous studies have been conducted on this topic[10, 123, 262, 402].
- COVID-19: COVID-19 is a contagious disease caused by the SARS-CoV-2 virus. It spreads through respiratory droplets and can cause symptoms such as fever, cough, and breathing difficulties. Severe cases may require hospitalization (cf. [127, 359]). Several papers have been published on the effective use of fuzzy graphs in relation to COVID-19 [212, 367, 387, 458].
- Communication networks: Communication networks are systems that allow the transfer of data or information between devices through interconnected nodes, using wired or wireless connections(cf.[315]). They include internet, phone, and satellite networks. Several papers have been published on the effective use of fuzzy graphs in relation to Communication networks[96, 232, 288, 289].
- SNS: SNS (Social Networking Service) is an online platform where users create profiles, share content, and connect with others (cf.[11, 131, 373]). Several papers have been published on the effective use of fuzzy graphs in relation to SNS[159, 230, 286].

2.4 Intuitionistic fuzzy graphs

An Intuitionistic Fuzzy Graph (IFG) extends fuzzy graphs by incorporating membership, non-membership, and hesitancy degrees for vertices and edges. An Intuitionistic Fuzzy Graphs have been the subject of extensive research [24,141,341,371]. To put it simply, it is a graph that represents the concept of Intuitionistic Fuzzy Sets (cf.[80–83, 165]) in graph form.

The Intuitionistic Fuzzy framework encompasses a wide range of mathematical and logical structures that have been extensively studied and developed in recent years. The primary concepts include:

1. Intuitionistic Fuzzy Set [455]
2. Intuitionistic Fuzzy Topological Spaces [130, 381]
3. Intuitionistic Fuzzy Logics [84]
4. Intuitionistic Fuzzy Algebra [22, 233]
5. Intuitionistic Fuzzy Environment [69, 277]

2.4.1 Definition of Intuitionistic Fuzzy Graph

The definition is provided below.

Definition 2.54 (Intuitionistic Fuzzy Graph (IFG))**.** [341] Let $G = (V, E)$ be a classical graph where V denotes the set of vertices and E denotes the set of edges. An *Intuitionistic Fuzzy Graph* (IFG) on G, denoted $G_{IF} = (A, B)$, is defined as follows:

1. (μ_A, ν_A) is an *Intuitionistic Fuzzy Set (IFS)* on the vertex set V. For each vertex $x \in V$, the degree of membership $\mu_A(x) \in [0, 1]$ and the degree of non-membership $\nu_A(x) \in [0, 1]$ satisfy:

$$\mu_A(x) + \nu_A(x) \leq 1$$

   The value $1 - \mu_A(x) - \nu_A(x)$ represents the hesitancy or uncertainty regarding the membership of x in the set.

2. (μ_B, ν_B) is an *Intuitionistic Fuzzy Relation (IFR)* on the edge set E. For each edge $(x, y) \in E$, the degree of membership $\mu_B(x, y) \in [0, 1]$ and the degree of non-membership $\nu_B(x, y) \in [0, 1]$ satisfy:

$$\mu_B(x, y) + \nu_B(x, y) \leq 1$$

   Additionally, the following constraints must hold for all $x, y \in V$:

$$\mu_B(x, y) \leq \mu_A(x) \wedge \mu_A(y)$$

$$\nu_B(x, y) \leq \nu_A(x) \vee \nu_A(y)$$

   In this definition:

- $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and non-membership of the vertex x, respectively.
- $\mu_B(x, y)$ and $\nu_B(x, y)$ represent the degree of membership and non-membership of the edge (x, y), respectively.
- If $\nu_A(x) = 0$ and $\nu_B(x, y) = 0$ for all $x \in V$ and $(x, y) \in E$, then the Intuitionistic Fuzzy Graph reduces to a Fuzzy Graph.

Example 2.55 (Intuitionistic Fuzzy Graph)**.** Consider the Intuitionistic Fuzzy Graph $G = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$, where $V = \{v_1, v_2, v_3, v_4\}$ and the edges $E = \{(v_1, v_2), (v_1, v_4), (v_2, v_3), (v_3, v_4), (v_2, v_4)\}$. The membership and non-membership degrees for the vertices are given as:

$$\mu_A(v_1) = 0.1, \quad \nu_A(v_1) = 0.4$$
$$\mu_A(v_2) = 0.3, \quad \nu_A(v_2) = 0.3$$
$$\mu_A(v_3) = 0.2, \quad \nu_A(v_3) = 0.4$$
$$\mu_A(v_4) = 0.4, \quad \nu_A(v_4) = 0.6$$

For the edges, the membership and non-membership degrees are given as follows:

- $\mu_B(v_1, v_2) = 0.1, \quad \nu_B(v_1, v_2) = 0.4$
- $\mu_B(v_1, v_4) = 0.1, \quad \nu_B(v_1, v_4) = 0.6$
- $\mu_B(v_2, v_3) = 0.1, \quad \nu_B(v_2, v_3) = 0.4$
- $\mu_B(v_3, v_4) = 0.1, \quad \nu_B(v_3, v_4) = 0.6$
- $\mu_B(v_2, v_4) = 0.3, \quad \nu_B(v_2, v_4) = 0.6$

This provides an example of how an Intuitionistic Fuzzy Graph can be structured, representing uncertainty in both vertex and edge membership using membership and non-membership degrees.

The following types of graphs are known for Intuitionistic Fuzzy Graphs.

Definition 2.56 (Strong Intuitionistic Fuzzy Graph)**.** [21] An Intuitionistic Fuzzy Graph $G = (A, B)$ is called a *strong intuitionistic fuzzy graph* if for all $xy \in E$:

$$\mu_B(xy) = \min(\mu_A(x), \mu_A(y)) \quad \text{and} \quad \nu_B(xy) = \max(\nu_A(x), \nu_A(y)).$$

Definition 2.57 (Complete Intuitionistic Fuzzy Graph)**.** (cf.[468]) An Intuitionistic Fuzzy Graph $G = (A, B)$ is called *complete* if for all $xy \in E$:

$$\mu_B(xy) = \min(\mu_A(x), \mu_A(y)) \quad \text{and} \quad \nu_B(xy) = \min(\nu_A(x), \nu_A(y)).$$

Proposition 2.58. *Every Intuitionistic Fuzzy Graph can be transformed into a Fuzzy Graph by restricting the non-membership function ν_A to 0 for all vertices.*

Proof. To transform G_{IF} into a fuzzy graph, we set the non-membership functions $\nu_A(x) = 0$ for all vertices $x \in V$, and $\nu_B(x, y) = 0$ for all edges $(x, y) \in E$.

With this restriction, the conditions simplify as follows:

$$\mu_A(x) + \nu_A(x) = \mu_A(x) \leq 1 \quad \text{for all } x \in V,$$

$$\mu_B(x, y) + \nu_B(x, y) = \mu_B(x, y) \leq 1 \quad \text{for all } (x, y) \in E.$$

Thus, G_{IF} reduces to a fuzzy graph $G_F = (\sigma, \mu)$, where:

- $\sigma : V \to [0, 1]$ is the membership function for vertices, corresponding to μ_A,
- $\mu : E \to [0, 1]$ is the membership function for edges, corresponding to μ_B,

and the constraints:

$$\mu(x, y) \leq \sigma(x) \wedge \sigma(y) \quad \text{for all } (x, y) \in E,$$

are satisfied as in the definition of a fuzzy graph. Therefore, by restricting the non-membership function ν_A to 0, we transform the Intuitionistic Fuzzy Graph into a Fuzzy Graph. □

2.4.2 Graph class for Intuitionistic Fuzzy Graph

We consider about Graph class for Intuitionistic Fuzzy Graph.

Theorem 2.59. *The following are examples of related graph classes, including but not limited to:*

- *Strong Intuitionistic Fuzzy Graphs [21]*
- *Perfect intuitionistic fuzzy graphs [188]*
- *Intuitionistic fuzzy competition graphs[388]*
- *Intuitionistic fuzzy threshold graphs[475]*
- *Balanced Intuitionistic Fuzzy Graphs[246]*
- *Bipolar Intuitionistic Fuzzy Competition Graphs[150]*
- *Intuitionistic Felicitous Fuzzy Graphs [94]*
- *Intuitionistic Fuzzy Soft Expert Graphs [464]*
- *Intuitionistic fuzzy tolerance graphs[389]*
- *Intuitionistic fuzzy planar graphs[63]*

- *Edge regular intuitionistic fuzzy graph[247]*
- *Edge Irregular Intuitionistic Fuzzy Graphs[331]*
- *m-Neighbourly Irregular Instuitionistic Fuzzy Graphs[291]*
- *R-edge regular intuitionistic fuzzy graphs[57]*
- *Perfectly Edge-Regular Intuitionistic Fuzzy Graphs[3]*
- *Edge Regular Intuitionistic Fuzzy M-Polargraphs[191]*
- *Intuitionistic fuzzy labeling graphs[386]*
- *Regular Interval-Valued Intuitionistic Fuzzy Graphs[2]*
- *Anti intuitionistic fuzzy graph[227]*
- *Interval-valued intuitionistic fuzzy graphs [268]*
- *Intuitionistic fuzzy directed hypergraphs[328]*
- *Complex intuitionistic fuzzy graphs[72]*
- *Complex t-Intuitionistic Fuzzy Graph[250]*
- *Intuitionistic fuzzy incidence graphs[334]*
- *Intuitionistic fuzzy k-partite hypergraphs[325, 326]*
- *Irregular Intuitionistic Fuzzy Graphs[308]*
- *intuitionistic L-fuzzy graph[446, 453]*
- *Bipolar intuitionistic anti fuzzy graphs[149]*
- *Intuitionistic anti-fuzzy graphs[324]*
- *intuitionistic product fuzzy graphs[456]*
- *intuitionistic k-partitioned fuzzy graph[345]*
- *intutionistic fuzzy multigraphs[85]*

Proof. Refer to each reference as needed. □

2.4.3 Application of intuitionistic fuzzy graph

We introduce the applications of intuitionistic fuzzy Graphs. Intuitionistic fuzzy Graphs hold potential for a wide range of applications. Below, we present an example of such applications.

- water supply systems: Water supply systems are networks that collect, treat, and distribute water from sources like rivers or reservoirs to homes and businesses(cf.[129, 273]). Several papers have explored the use of Intuitionistic fuzzy Graphs in water supply systems[251, 404].
- cellular network: A cellular network is a communication system that uses multiple cell towers to provide wireless connectivity for mobile devices(cf.[482]). Several papers have explored the use of Intuitionistic fuzzy Graphs in cellular network[476].
- COVID-19: Several papers have explored the use of Intuitionistic fuzzy Graphs in COVID-19[451]

2.5 Neutrosophic Graph

We present the concept of a neutrosophic graph [41, 115, 200, 215, 385, 436], which builds upon and expands the theory of fuzzy graphs [237]. Neutrosophic graphs offer a powerful tool for addressing uncertainty, making them particularly useful in diverse domains such as social networks and industrial systems. While fuzzy graph theory has already proven its relevance in modern scientific and technological applications, including operations research, neural networks [4, 283], artificial intelligence [4, 283], and decision-making [237], neutrosophic graphs provide an enhanced framework for more complex scenarios.yaqoob2019complex

The Neutrosophic framework encompasses a wide range of mathematical and logical structures that have been extensively studied and developed in recent years. The primary concepts include:

1. Neutrosophic Set [110, 166, 419, 421, 438, 471]
2. Neutrosophic Offset[179, 418, 423, 431, 432]
3. Neutrosophic Topological Spaces [76, 390]
4. Neutrosophic Logics [78, 375, 419]

5. Neutrosophic Algebraic Structures [47, 242]
6. Neutrosophic Environment [384, 437]
7. NeutroGeometry[430]
8. Neutrosophic Statistics[64, 423]
9. Neutrosophic Physics[420]
10. Neutrosophic Sociology[427]
11. Neutrosophic Automata[184]

The definitions are provided below.

Definition 2.60. [116,436] A neutrosophic graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ is defined as a graph where $\sigma_i : V \to [0, 1]$, $\mu_i : E \to [0, 1]$, and for every $v_i v_j \in E$, the following condition holds: $\mu(v_i v_j) \leq \sigma(v_i) \wedge \sigma(v_j)$.

1. σ is called the neutrosophic vertex set.
2. μ is called the neutrosophic edge set.
3. $|V|$ is called the order of NTG, and it is denoted by $O(NTG)$.
4. $\sum_{v \in V} \sigma(v)$ is called the neutrosophic order of NTG, and it is denoted by $On(NTG)$.
5. $|E|$ is called the size of NTG, and it is denoted by $S(NTG)$.
6. $\sum_{e \in E} \mu(e)$ is called the neutrosophic size of NTG, and it is denoted by $Sn(NTG)$.

The Examples of neutrosophic graph is following.

Example 2.61. (cf.[116]) Consider a neutrosophic graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ with four vertices $V = \{v_1, v_2, v_3, v_4\}$, as shown in the diagram.

The neutrosophic membership degrees of the vertices are as follows:

$$\sigma(v_1) = (0.5, 0.1, 0.4), \quad \sigma(v_2) = (0.6, 0.3, 0.2),$$
$$\sigma(v_3) = (0.2, 0.3, 0.4), \quad \sigma(v_4) = (0.4, 0.2, 0.5)$$

The neutrosophic membership degrees of the edges are as follows:

$$\mu(v_1 v_2) = (0.2, 0.3, 0.4), \quad \mu(v_2 v_3) = (0.3, 0.3, 0.4),$$
$$\mu(v_3 v_4) = (0.2, 0.3, 0.4), \quad \mu(v_4 v_1) = (0.1, 0.2, 0.5)$$

In this case, the neutrosophic graph NTG has the following properties:

- Vertices v_1, v_2, v_3, v_4 are connected by edges with varying neutrosophic membership degrees.
- The neutrosophic relations ensure that for every edge $v_i v_j \in E$, $\mu(v_i v_j) \leq \sigma(v_i) \wedge \sigma(v_j)$, where $\wedge$ denotes the minimum operation.

Proposition 2.62. *(cf.[436]) Every neutrosophic Graph can be transformed into a fuzzy graph.*

Proof. To transform a neutrosophic graph $G_N = (V, E, T, I, F)$ into a fuzzy graph, we reduce the multiple components (truth T, indeterminacy I, and falsity F) into a single membership function.

Define the fuzzy graph $G_F = (V, E, \mu)$ where $\mu : V \times V \to [0, 1]$ is the membership function, given by:

$$\mu(u, v) = T(u, v) - I(u, v) - F(u, v),$$

subject to $\mu(u, v) \in [0, 1]$. This transformation subtracts the indeterminacy and falsity values from the truth value to produce a single fuzzy membership value.

- If $I(u, v) = F(u, v) = 0$, then $\mu(u, v) = T(u, v)$, reflecting a pure truth-based fuzzy graph.

- If $T(u, v) = 0$, then $\mu(u, v) = 0$, meaning no edge exists.
- If $I(u, v) + F(u, v) \geq T(u, v)$, then $\mu(u, v) = 0$, indicating that the edge is too uncertain or false.

Thus, the transformation from a neutrosophic graph to a fuzzy graph is valid. □

Proposition 2.63. *Every Neutrosophic Graph can be transformed into an Intuitionistic Fuzzy Graph by setting the indeterminacy value to zero.*

Proof. Let $G_N = (V, E, T, I, F)$ be a neutrosophic graph, where:

- $T : V \to [0, 1]$ is the truth-membership function for vertices,
- $I : V \to [0, 1]$ is the indeterminacy-membership function for vertices,
- $F : V \to [0, 1]$ is the falsity-membership function for vertices,
- Similarly, $T : E \to [0, 1]$, $I : E \to [0, 1]$, and $F : E \to [0, 1]$ are the truth, indeterminacy, and falsity functions for edges.

To transform G_N into an Intuitionistic Fuzzy Graph $G_{IF} = (V, E, \mu_A, v_A, \mu_B, v_B)$, we make the following assignments:

- For vertices:
$$\mu_A(v) = T(v), \quad v_A(v) = F(v),$$
where $\mu_A(v)$ represents the membership degree of vertex v, and $v_A(v)$ represents the non-membership degree. The indeterminacy value $I(v)$ in the neutrosophic graph is set to zero, implying no uncertainty.
- For edges:
$$\mu_B(e) = T(e), \quad v_B(e) = F(e),$$
where $\mu_B(e)$ represents the membership degree of edge e, and $v_B(e)$ represents the non-membership degree.

Since the indeterminacy function $I(v)$ and $I(e)$ is set to zero, the hesitation degree (indeterminacy) vanishes, reducing the neutrosophic graph to an Intuitionistic Fuzzy Graph. □

2.5.1 Graph class related to neutrosophic graph

We describe the graph classes related to neutrosophic graphs. Due to the applicability and significance of neutrosophic graphs, many related graph classes have been extensively studied. The definitions are provided below.

Theorem 2.64. *The following are examples of related Neutrosophic graph classes, including but not limited to:*

- *Bipolar Neutrosophic Graphs [39]*
- *Interval-Valued Neutrosophic Graphs [409]*
- *Single-Valued Neutrosophic Graphs [43, 116]*
- *Interval Complex Neutrosophic Graphs [109, 220]*
- *Neutrosophic Hypergraphs [190]*
- *Neutrosophic Vague Line Graphs [220]*
- *Neutrosophic Vague Graphs [223]*
- *Single-Valued Neutrosophic Signed Graphs [306]*
- *Neutrosophic Soft Rough Graphs [33]*
- *Neutrosophic Incidence Graphs [40]*
- *Fermatean Neutrosophic Dombi Fuzzy Graphs [397]*
- *Interval valued pentapartitioned neutrosophic graphs [109]*
- *Single valued pentapartitioned neutrosophic graphs [140]*
- *pentapartitioned neutrosophic graphs[363]*
- *t-Neutrosophic Fuzzy graph[248]*

- *Complex t-Neutrosophic Graph[252]*
- *Regular neutrosophic graphs[215]*
- *Q-neutrosophic soft graphs[462]*
- *Balanced Neutrosophic Graphs[416]*
- *Neutrosophic minimum spanning tree graph [279]*
- *HyperNeutrosophic Graph[177]*
- *SuperHyperNeutrosophic Graph[177]*

Proof. Refer to each reference as needed. □

2.5.2 Application of neutrosophic graph

We introduce the applications of Neutrosophic Graphs. Neutrosophic Graphs, which can generalize fuzzy graphs, hold potential for a wide range of applications. Below, we present an example of such applications.

- Decision-making: Several papers have explored the use of Neutrosophic Graphs in Decision-making[38, 113, 465, 480].
- COVID-19: Several papers have explored the use of Neutrosophic Graphs in COVID-19 countermeasures[363]
- Hospital Infrastructure Design: A paper has explored the use of Neutrosophic Graphs in Hospital Infrastructure Design[62].
- Internet Streaming Services: Internet streaming services deliver digital content like video, music, or live broadcasts to users via the internet in real-time (cf.[474]). The study is conducted in [249].
- Natural disaster: A natural disaster is a catastrophic event caused by natural processes, such as earthquakes, tsunamis, hurricanes, or floods, leading to widespread damage and loss of life (cf.[60, 338]). In the study [58], the potential application of Neutrosophic Graphs in Japan's earthquake response is being explored. Additionally, [407] examines response methods for tsunami management in Japan.
- Networks: Several papers have explored the use of Neutrosophic Graphs in mobile network[117] and wireless networks[114, 205].

2.6 Turiyam Neutrosophic Graph

Research on Turiyam Neutrosophic Graphs, which incorporate parameters into Neutrosophic Graphs, is currently being conducted [185–187]. These graphs are a graphical representation of the Turiyam Neutrosophic Set [410]. Similar concepts include four-valued logic [91, 125]. The definition is provided below.

Definition 2.65 (Turiyam Neutrosophic Graph)**.** [185–187] Let $G = (V, E)$ be a classical graph with a finite set of vertices $V = \{v_i : i = 1, 2, \ldots, n\}$ and edges $E = \{(v_i, v_j) : i, j = 1, 2, \ldots, n\}$. A *Turiyam Neutrosophic Graph* of G, denoted $G^T = (V^T, E^T)$, is defined as follows:

1. **Turiyam Neutrosophic Vertex Set**: For each vertex $v_i \in V$, the Turiyam Neutrosophic graph assigns the following mappings:
$$t(v_i), iv(v_i), fv(v_i), lv(v_i) : V \to [0, 1],$$
where:
   - $t(v_i)$ is the truth value (tv) of the vertex v_i,
   - $iv(v_i)$ is the indeterminacy value (iv) of v_i,
   - $fv(v_i)$ is the falsity value (fv) of v_i,
   - $lv(v_i)$ is the Turiyam Neutrosophic state (or liberal value) (lv) of v_i,

   for all $v_i \in V$, such that the following condition holds for each vertex:
$$0 \leq t(v_i) + iv(v_i) + fv(v_i) + lv(v_i) \leq 4.$$

2. **Turiyam Neutrosophic Edge Set**: For each edge $(v_i, v_j) \in E$, the Turiyam Neutrosophic graph assigns the following mappings:

$$t(v_i, v_j), iv(v_i, v_j), fv(v_i, v_j), lv(v_i, v_j) : E \to [0, 1],$$

where:

- $t(v_i, v_j)$ is the truth value of the edge (v_i, v_j),
- $iv(v_i, v_j)$ is the indeterminacy value of (v_i, v_j),
- $fv(v_i, v_j)$ is the falsity value of (v_i, v_j),
- $lv(v_i, v_j)$ is the Turiyam Neutrosophic state (or liberal value) of (v_i, v_j),

for all $(v_i, v_j) \in E$, such that the following condition holds for each edge:

$$0 \leq t(v_i, v_j) + iv(v_i, v_j) + fv(v_i, v_j) + lv(v_i, v_j) \leq 4.$$

In this case, V^T represents the Turiyam Neutrosophic vertex set of the graph G^T, and E^T represents the Turiyam Neutrosophic edge set of G^T.

Proposition 2.66. *A Turiyam Neutrosophic graph is a generalization of the fuzzy graph.*

Proof. Let $G = (V, E)$ be a Turiyam Neutrosophic graph, where each vertex and edge is associated with four membership values: truth-membership (t), indeterminacy-membership (i), falsity-membership (f), and Turiyam Neutrosophic state-membership (l).

To reduce a Turiyam Neutrosophic graph to a fuzzy graph, we set the values of the indeterminacy-membership, falsity-membership, and Turiyam Neutrosophic state-membership to zero, keeping only the truth-membership value. This corresponds to the membership function used in a fuzzy graph.

For each vertex $v \in V$ and edge $e \in E$, we set the following:

- Indeterminacy-membership $i(v), i(e) = 0$,
- Falsity-membership $f(v), f(e) = 0$,
- Turiyam Neutrosophic state-membership $l(v), l(e) = 0$.

As a result, only the truth-membership value $t(v)$ for each vertex v and $t(e)$ for each edge e remains. Since a fuzzy graph is characterized by a single membership function μ that assigns values between 0 and 1 to each vertex and edge, this reduced form of the Turiyam Neutrosophic graph is equivalent to a fuzzy graph where:

$$\mu(v) = t(v) \quad \text{and} \quad \mu(e) = t(e).$$

Thus, by eliminating the indeterminacy, falsity, and Turiyam Neutrosophic state values, a Turiyam Neutrosophic graph reduces to a fuzzy graph. Therefore, a Turiyam Neutrosophic graph is a generalization of a fuzzy graph. □

Proposition 2.67. *A Turiyam Neutrosophic graph is a generalization of the neutrosophic graph.*

Proof. Let $G_N = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ be a neutrosophic graph, where:

- σ_1, σ_2, and σ_3 are the truth, indeterminacy, and falsity membership functions on the vertices, respectively, with $\sigma_i : V \to [0, 1]$.
- μ_1, μ_2, and μ_3 are the truth, indeterminacy, and falsity membership functions on the edges, respectively, with $\mu_i : E \to [0, 1]$.

The neutrosophic graph assigns three membership degrees to each vertex and edge, namely truth, indeterminacy, and falsity.

A Turiyam Neutrosophic graph $G^T = (V^T, E^T)$ extends the neutrosophic graph by adding a fourth component, the "Turiyam Neutrosophic state" or liberal value, to both vertices and edges. In a Turiyam Neutrosophic graph, for each vertex $v \in V$ and each edge $e \in E$, the following values are assigned:

$$t(v), iv(v), fv(v), lv(v) \quad \text{for each vertex } v \in V,$$

$$t(e), iv(e), fv(e), lv(e) \quad \text{for each edge } e \in E,$$

where:

- $t(v), iv(v), fv(v)$ correspond to the truth, indeterminacy, and falsity values of the vertex v, matching the components of a neutrosophic graph.
- $lv(v)$ is an additional component, representing the Turiyam Neutrosophic state (liberal value) of the vertex.
- Similarly, $t(e), iv(e), fv(e)$ correspond to the truth, indeterminacy, and falsity values of the edge e, and $lv(e)$ is the Turiyam Neutrosophic state of the edge.

Thus, the Turiyam Neutrosophic graph includes the structure of the neutrosophic graph but adds an extra dimension of information with the lv component, making it a more general structure. The conditions for the vertex set V^T and edge set E^T in the Turiyam Neutrosophic graph are:

$$0 \leq t(v) + iv(v) + fv(v) + lv(v) \leq 4 \quad \text{for each vertex } v,$$

$$0 \leq t(e) + iv(e) + fv(e) + lv(e) \leq 4 \quad \text{for each edge } e.$$

By setting $lv(v) = 0$ and $lv(e) = 0$ for all vertices and edges, the Turiyam Neutrosophic graph reduces to a neutrosophic graph where only the truth, indeterminacy, and falsity components are retained. Therefore, a neutrosophic graph can be viewed as a special case of a Turiyam Neutrosophic graph, where the liberal value is not considered.

This demonstrates that a Turiyam Neutrosophic graph is a generalization of the neutrosophic graph, as it extends the latter by adding an additional component, lv, while retaining the structure of truth, indeterminacy, and falsity. □

2.7 Single-Valued Neutrosophic Graph

A *Single-Valued Neutrosophic Graph* (SVNG) is a generalization of classical graphs, where each vertex and edge is characterized by degrees of truth, indeterminacy, and falsity, taking values from the interval $[0, 1]$ [284]. The definition is provided below.

Definition 2.68. [46, 244, 284] A *Single-Valued Neutrosophic Graph* G is defined as a pair $G = (A, B)$, where:

- $A : V \to [0, 1]$ is a single-valued neutrosophic set on the vertex set V,
- $B : V \times V \to [0, 1]$ is a single-valued neutrosophic relation on the edge set $E \subseteq V \times V$.

For each pair of vertices $x, y \in V$, the following conditions hold:

$$T_B(xy) \leq \min\{T_A(x), T_A(y)\},$$

$$I_B(xy) \leq \min\{I_A(x), I_A(y)\},$$

$$F_B(xy) \leq \max\{F_A(x), F_A(y)\},$$

where:

- $T_A(x), I_A(x)$, and $F_A(x)$ represent the truth-membership, indeterminacy-membership, and falsity-membership values of vertex x,
- $T_B(xy)$, $I_B(xy)$, and $F_B(xy)$ represent the truth-membership, indeterminacy-membership, and falsity-membership values of edge xy.
- **Single-Valued Neutrosophic Vertex Set** A: This defines the neutrosophic membership functions for the vertices. Each vertex $x \in V$ is associated with three membership degrees: truth ($T_A(x)$), indeterminacy ($I_A(x)$), and falsity ($F_A(x)$).
- **Single-Valued Neutrosophic Edge Set** B: This defines the neutrosophic membership functions for the edges. Each edge $xy \in E$ has corresponding neutrosophic membership degrees: truth ($T_B(xy)$), indeterminacy ($I_B(xy)$), and falsity ($F_B(xy)$).
- If the neutrosophic relation B is symmetric, the graph $G = (A, B)$ is considered an undirected single-valued neutrosophic graph.
- If the relation B is not symmetric, the graph $G = (A, B)$ is a directed single-valued neutrosophic graph.

2.7.1 Graph class related to single-valued neutrosophic graph

We introduce related graph classes of the single-valued neutrosophic graph. The following properties hold.

Theorem 2.69. *The following are examples of related Neutrosophic graph classes, including but not limited to:*

- *Single-Valued Neutrosophic Graphs [43, 116]*
- *Single-Valued Neutrosophic Signed Graphs [306]*
- *Single valued neutrosophic signed digraph[415]*
- *Single valued pentapartitioned neutrosophic graphs[140]*
- *Isolated single valued neutrosophic graphs[111]*
- *Bipolar Single Valued Neutrosophic Isolated Graphs[108]*
- *Uniform single valued neutrosophic graphs[112]*
- *Accessible single-valued neutrosophic graphs[206]*
- *single-valued co-neutrosophic graphs[154]*
- *Single-valued neutrosophic line graphs[332]*
- *Regular single valued neutrosophic hypergraphs[298]*
- *KM-single valued neutrosophic metric graphs[207]*
- *antipodal single valued neutrosophic graph[295]*
- *Regular bipolar single valued neutrosophic hypergraphs[297]*
- *Self-Centered Single Valued Neutrosophic Graphs[469]*

2.8 Single-Valued Neutrosophic Hypergraph

A *Single-Valued Neutrosophic Hypergraph* (SVNH) is a generalization of a classical hypergraph, incorporating neutrosophic sets to account for degrees of truth, indeterminacy, and falsity between vertices and edges[30, 31, 43, 296].

Definition 2.70. [43] Let $V = \{v_1, v_2, \ldots, v_n\}$ be a finite set of vertices, and let $E = \{E_1, E_2, \ldots, E_m\}$ be a finite family of single-valued neutrosophic subsets of V. Each hyperedge $E_i \in E$ is represented as:

$$E_i = \left\{ \left(v_j, T_{E_i}(v_j), I_{E_i}(v_j), F_{E_i}(v_j)\right) \mid v_j \in V \right\}$$

where:

- $T_{E_i}(v_j)$ is the truth-membership degree of vertex v_j in hyperedge E_i,
- $I_{E_i}(v_j)$ is the indeterminacy-membership degree of vertex v_j in hyperedge E_i,
- $F_{E_i}(v_j)$ is the falsity-membership degree of vertex v_j in hyperedge E_i.

The pair $H = (V, E)$ is called a *single-valued neutrosophic hypergraph*, where:

- V is the set of vertices, and
- E is the family of single-valued neutrosophic hyperedges.
- **Adjacency of Vertices:** Two vertices u and v are adjacent if they belong to the same hyperedge E_i, i.e., $u, v \in \text{supp}(E_i)$.
- **Connectivity:** A single-valued neutrosophic hypergraph is connected if every pair of vertices can be connected by a sequence of adjacent vertices.
- **Adjacency of Edges:** Two hyperedges E_i and E_j are adjacent if they share at least one common vertex, i.e., $\text{supp}(E_i) \cap \text{supp}(E_j) \neq \emptyset$.
- **Order and Size:**
  - The *order* of the hypergraph is $|V|$, the number of vertices.
  - The *size* of the hypergraph is $|E|$, the number of hyperedges.

- **Cardinality of a Hyperedge:** The degree of a hyperedge E_i is the sum of the truth-membership, indeterminacy-membership, and falsity-membership values of the vertices in that hyperedge:

$$d_H(E_i) = \sum_{v_j \in E_i} \left(T_{E_i}(v_j) + I_{E_i}(v_j) + F_{E_i}(v_j)\right)$$

- **Rank and Anti-Rank:**
  - The *rank* of the hypergraph is the maximum cardinality of any hyperedge: $\max_{E_i \in E} d_H(E_i)$.
  - The *anti-rank* of the hypergraph is the minimum cardinality of any hyperedge: $\min_{E_i \in E} d_H(E_i)$.
- **Linear Single-Valued Neutrosophic Hypergraph:** The hypergraph is called *linear* if any two distinct vertices belong to at most one common hyperedge, i.e.,

$$|\text{supp}(E_i) \cap \text{supp}(E_j)| \leq 1 \quad \text{for all } E_i, E_j \in E.$$

- **Membership Constraint:** For each vertex v_j in any hyperedge E_i, the following constraint must hold:

$$0 \leq T_{E_i}(v_j) + I_{E_i}(v_j) + F_{E_i}(v_j) \leq 3$$

  This ensures that the sum of the truth, indeterminacy, and falsity values for any vertex in a hyperedge is valid.
- **Linear Hypergraphs:** The linear property ensures that any pair of vertices belongs to at most one common hyperedge, which preserves the structure of the hypergraph.

Example 2.71. Consider a single-valued neutrosophic hypergraph $H = (V, E)$ such that:

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}, \quad E = \{E_1, E_2, E_3, E_4, E_5, E_6\},$$

where:

$$E_1 = \{(v_1, 0.3, 0.4, 0.6), (v_3, 0.7, 0.4, 0.4)\},$$
$$E_2 = \{(v_1, 0.3, 0.4, 0.6), (v_2, 0.5, 0.7, 0.6)\},$$
$$E_3 = \{(v_2, 0.5, 0.7, 0.6), (v_4, 0.6, 0.4, 0.8)\},$$
$$E_4 = \{(v_3, 0.7, 0.4, 0.4), (v_6, 0.4, 0.2, 0.7)\},$$
$$E_5 = \{(v_3, 0.7, 0.4, 0.4), (v_5, 0.6, 0.7, 0.5)\},$$
$$E_6 = \{(v_5, 0.6, 0.7, 0.5), (v_6, 0.4, 0.2, 0.7)\},$$

and:

$$E_7 = \{(v_4, 0.6, 0.4, 0.8), (v_6, 0.4, 0.2, 0.7)\}.$$

2.9 Known Result

Based on the previous discussions, the following theorem can be understood.

Theorem 2.72. *In each graph class, the following relationships hold.*

- *An empty graph and a null graph can be represented as 2-valued graphs and 3-valued graphs.*
- *Every edge-fuzzy graph can be transformed into a 2-valued graph by thresholding the edge membership values.*
- *Every fuzzy graph can be transformed into a 3-valued graph by mapping the fuzzy membership values of vertices and edges to the values {-1, 0, 1}.*
- *Every Intuitionistic Fuzzy Graph can be transformed into a Fuzzy Graph by restricting the non-membership function v_A to 0 for all vertices.*
- *Every Neutrosophic Graph can be transformed into an Intuitionistic Fuzzy Graph by setting the indeterminacy value to zero.*

3. New graph class definition and Result of this paper

The results in this paper are presented as follows.

3.1 General intuitionic fuzzy graph

Following the examples of the General Fuzzy Graph and General Neutrosophic Graph [173, 336], we introduce the definition of the General Intuitionistic Fuzzy Graph. The definition is provided below.

Definition 3.1 (General Intuitionistic Fuzzy Graph)**.** A *general intuitionistic fuzzy graph* with V as the underlying set is defined as a pair of functions $G_{IF} = (\mu_A, \nu_A, \mu_B, \nu_B)$, where:

- $\mu_A : V \to [0, 1]$ is an intuitionistic fuzzy subset of V representing the membership degree of vertices, and $\nu_A : V \to [0, 1]$ represents the non-membership degree of vertices.
- $\mu_B : V \times V \to [0, 1]$ is an intuitionistic fuzzy relation on $V \times V$ representing the membership degree of edges, and $\nu_B : V \times V \to [0, 1]$ represents the non-membership degree of edges.

This definition generalizes the traditional intuitionistic fuzzy graph by allowing the membership and non-membership values of edges to be independent of the membership and non-membership values of their incident vertices.

Definition 3.2 (Weak General Intuitionistic Fuzzy Graph)**.** A *weak general intuitionistic fuzzy graph* with V as the underlying set is defined as a pair of functions $G_{IF} = (\mu_A, \nu_A, \mu_B, \nu_B)$, where:

- $\mu_A : V \to [0, 1]$ is an intuitionistic fuzzy subset of V representing the membership degree of vertices, and $\nu_A : V \to [0, 1]$ represents the non-membership degree of vertices.
- $\mu_B : V \times V \to [0, 1]$ is an intuitionistic fuzzy relation on $V \times V$ representing the membership degree of edges, and $\nu_B : V \times V \to [0, 1]$ represents the non-membership degree of edges.

Additionally, for all $(u, v) \in E$, the membership degree of the edge $\mu_B(u, v)$ satisfies:

$$\mu_B(u, v) \neq \mu_A(u) \wedge \mu_A(v),$$

where $\wedge$ denotes the minimum operation.

This condition implies that the membership degree of an edge is not strictly determined by the membership degrees of its vertices.

The relationships with other graph classes are described below.

Theorem 3.3. *Every intuitionistic fuzzy graph is a weak general intuitionistic fuzzy graph.*

Proof. Let $G_{IF} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$ be an intuitionistic fuzzy graph. In an intuitionistic fuzzy graph, for each edge $(u, v) \in E$, the membership degree $\mu_B(u, v)$ is given by:

$$\mu_B(u, v) = \mu_A(u) \wedge \mu_A(v),$$

where $\wedge$ denotes the minimum operation.

In a weak general intuitionistic fuzzy graph, the condition required is $\mu_B(u, v) \neq \mu_A(u) \wedge \mu_A(v)$, but this does not prevent $\mu_B(u, v)$ from satisfying $\mu_B(u, v) = \mu_A(u) \wedge \mu_A(v)$ in specific cases. Thus, every intuitionistic fuzzy graph meets the weaker condition required by a weak general intuitionistic fuzzy graph.

Hence, every intuitionistic fuzzy graph is a weak general intuitionistic fuzzy graph. □

Theorem 3.4. *Every weak general intuitionistic fuzzy graph is a general intuitionistic fuzzy graph.*

Proof. By the definition of a weak general intuitionistic fuzzy graph, for all vertices $u, v \in V$ and edges $(u, v) \in E$, the following condition holds:

$$\mu_B(u, v) \neq \mu_A(u) \wedge \mu_A(v),$$

where $\wedge$ denotes the minimum operation.

Now, consider the definition of a general intuitionistic fuzzy graph. A general intuitionistic fuzzy graph $G = (\mu_A, \nu_A, \mu_B, \nu_B)$ allows the membership values of the edges $\mu_B(u, v)$ to take any value independently of the membership values of the vertices.

Since the condition $\mu_B(u, v) \neq \mu_A(u) \wedge \mu_A(v)$ is less restrictive than requiring $\mu_B(u, v) = \mu_A(u) \wedge \mu_A(v)$, any weak general intuitionistic fuzzy graph satisfies the conditions of a general intuitionistic fuzzy graph.

Thus, every weak general intuitionistic fuzzy graph is a general intuitionistic fuzzy graph. □

Theorem 3.5. *Every weak general intuitionistic fuzzy graph can be transformed into a weak general fuzzy graph by setting the non-membership functions to zero.*

Proof. Let $G_{IF} = (\mu_A, \nu_A, \mu_B, \nu_B)$ be a weak general intuitionistic fuzzy graph. To transform G_{IF} into a weak general fuzzy graph, set $\nu_A(v) = 0$ and $\nu_B(u, v) = 0$ for all vertices $v \in V$ and edges $(u, v) \in E$.

This results in a weak general fuzzy graph $G_F = (\sigma, \mu)$ where:

$$\sigma(v) = \mu_A(v) \quad \text{and} \quad \mu(u, v) = \mu_B(u, v).$$

The condition $\mu_B(u, v) \neq \mu_A(u) \wedge \mu_A(v)$ in the weak general intuitionistic fuzzy graph is preserved as $\mu(u, v) \neq \sigma(u) \wedge \sigma(v)$ in the weak general fuzzy graph.

Thus, every weak general intuitionistic fuzzy graph can be transformed into a weak general fuzzy graph. □

Corollary 3.6. *Every general intuitionistic fuzzy graph can be transformed into a general fuzzy graph by setting the non-membership functions to zero.*

Proof. It can be proven in the same way as above. □

3.2 Single-Valued Turiyam Neutrosophic Graph

A *Single-Valued Turiyam Neutrosophic Graph* (SVTG) is a generalization of classical graphs and Single-Valued Neutrosophic Graphs, where each vertex and edge is characterized by four degrees: truth, indeterminacy, falsity, and Turiyam Neutrosophic (liberal) state, taking values from the interval $[0, 1]$. Note that a Single-Valued Turiyam Neutrosophic Graph can be generalized within the framework of a Single-Valued Quadripartitioned Neutrosophic Graph (cf. [417])

Definition 3.7. A *Single-Valued Turiyam Neutrosophic Graph* G is defined as a pair $G = (A, B)$, where:

- $A : V \to [0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$ is a single-valued Turiyam Neutrosophic set on the vertex set V, assigning four degrees to each vertex: truth $t(v)$, indeterminacy $iv(v)$, falsity $f(v)$, and Turiyam Neutrosophic state $lv(v)$.
- $B : V \times V \to [0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$ is a single-valued Turiyam Neutrosophic relation on the edge set $E \subseteq V \times V$, assigning four degrees to each edge: truth $t(v_i, v_j)$, indeterminacy $iv(v_i, v_j)$, falsity $f(v_i, v_j)$, and Turiyam Neutrosophic state $lv(v_i, v_j)$.

For each pair of vertices $v_i, v_j \in V$, the following conditions must hold:

$$t(v_i, v_j) \leq \min\{t(v_i), t(v_j)\},$$

$$iv(v_i, v_j) \leq \min\{iv(v_i), iv(v_j)\},$$

$$f(v_i, v_j) \leq \max\{f(v_i), f(v_j)\},$$

$$lv(v_i, v_j) \leq \min\{lv(v_i), lv(v_j)\}.$$

In addition, for any vertex $v \in V$, the following constraint applies:

$$0 \leq t(v) + iv(v) + f(v) + lv(v) \leq 4.$$

For any edge $(v_i, v_j) \in E$, the corresponding constraint is:

$$0 \leq t(v_i, v_j) + iv(v_i, v_j) + f(v_i, v_j) + lv(v_i, v_j) \leq 4.$$

Example 3.8. Consider a graph G with three vertices V_1, V_2, V_3 and three edges connecting them. Each vertex and edge is assigned four values representing the truth (t), indeterminacy (iv), falsity (f), and Turiyam Neutrosophic state (lv) values.

Let the vertex set $V = \{V_1, V_2, V_3\}$ be defined as follows:

$$V_1 = (t(V_1), iv(V_1), f(V_1), lv(V_1)) = (0.3, 0.4, 0.2, 0.5)$$

$$V_2 = (t(V_2), iv(V_2), f(V_2), lv(V_2)) = (0.2, 0.3, 0.3, 0.4)$$

$$V_3 = (t(V_3), iv(V_3), f(V_3), lv(V_3)) = (0.4, 0.4, 0.3, 0.4)$$

The edges connecting these vertices are given as:

$$(V_1, V_2) = (t(V_1, V_2), iv(V_1, V_2), f(V_1, V_2), lv(V_1, V_2)) = (0.2, 0.2, 0.3, 0.3)$$

$$(V_2, V_3) = (t(V_2, V_3), iv(V_2, V_3), f(V_2, V_3), lv(V_2, V_3)) = (0.3, 0.2, 0.2, 0.3)$$

$$(V_1, V_3) = (t(V_1, V_3), iv(V_1, V_3), f(V_1, V_3), lv(V_1, V_3)) = (0.3, 0.3, 0.2, 0.2)$$

Each vertex has been assigned values for truth, indeterminacy, falsity, and Turiyam Neutrosophic states, which satisfy the constraint:

$$0 \leq t(v) + iv(v) + f(v) + lv(v) \leq 4 \quad \forall v \in \{V_1, V_2, V_3\}.$$

- For V_1, we have $0.3 + 0.4 + 0.2 + 0.5 = 1.4$.
- For V_2, $0.2 + 0.3 + 0.3 + 0.4 = 1.2$.
- For V_3, $0.4 + 0.4 + 0.3 + 0.4 = 1.5$.

These satisfy the constraint for the vertex membership condition.

Similarly, each edge has four values assigned:

$$0 \leq t(v_i, v_j) + iv(v_i, v_j) + f(v_i, v_j) + lv(v_i, v_j) \leq 4 \quad \forall (v_i, v_j) \in \{(V_1, V_2), (V_2, V_3), (V_1, V_3)\}.$$

- For (V_1, V_2), we have $0.2 + 0.2 + 0.3 + 0.3 = 1.0$.
- For (V_2, V_3), $0.3 + 0.2 + 0.2 + 0.3 = 1.0$.
- For (V_1, V_3), $0.3 + 0.3 + 0.2 + 0.2 = 1.0$.

These satisfy the edge membership condition.

Theorem 3.9. *Every Single-Valued Turiyam Neutrosophic Graph $G^T = (V^T, E^T)$, where the vertices and edges have associated truth (t), indeterminacy (iv), falsity (f), and Turiyam Neutrosophic (lv) values, can be transformed into a Single-Valued Neutrosophic Graph $G^N = (V^N, E^N)$ by merging the Turiyam Neutrosophic value lv into the truth, indeterminacy, and falsity components of the graph, while maintaining the membership constraints $t + iv + f \leq 3$.*

Proof. For each vertex $v \in V^T$, and each edge $e \in E^T$, redistribute the Turiyam Neutrosophic value lv as follows:

$$t'(v) = t(v) + \alpha \cdot lv(v),$$
$$iv'(v) = iv(v) + \beta \cdot lv(v),$$
$$f'(v) = f(v) + \gamma \cdot lv(v),$$

where $\alpha, \beta, \gamma \in [0, 1]$ are constants such that $\alpha + \beta + \gamma = 1$. These constants determine how the Turiyam Neutrosophic value is distributed between the truth, indeterminacy, and falsity components.

After redistributing the Turiyam Neutrosophic value, the new membership values for vertices and edges must satisfy the Single-Valued Neutrosophic constraint:

$$0 \leq t'(v) + iv'(v) + f'(v) \leq 3.$$

By construction, the redistribution satisfies this condition because:

$$t'(v) + iv'(v) + f'(v) = (t(v) + \alpha \cdot lv(v)) + (iv(v) + \beta \cdot lv(v)) + (f(v) + \gamma \cdot lv(v)).$$

Simplifying:

$$t'(v) + iv'(v) + f'(v) = t(v) + iv(v) + f(v) + (\alpha + \beta + \gamma) \cdot lv(v).$$

Since $\alpha + \beta + \gamma = 1$, we have:

$$t'(v) + iv'(v) + f'(v) = t(v) + iv(v) + f(v) + lv(v).$$

From the original Turiyam Neutrosophic condition, we know that:

$$0 \leq t(v) + iv(v) + f(v) + lv(v) \leq 4.$$

Thus, after removing the Turiyam Neutrosophic component $lv(v)$, we have:

$$0 \leq t'(v) + iv'(v) + f'(v) \leq 3.$$

The same process applies to edges, ensuring that for each edge $e \in E^T$:

$$0 \leq t'(e) + iv'(e) + f'(e) \leq 3.$$

The result of the redistribution of the Turiyam Neutrosophic value is that the modified graph $G^N = (V^N, E^N)$ satisfies the conditions for a Single-Valued Neutrosophic Graph, with no Turiyam Neutrosophic values remaining.

Thus, the Single-Valued Turiyam Neutrosophic Graph G^T has been successfully transformed into a Single-Valued Neutrosophic Graph G^N, where:

- $t'(v), iv'(v), f'(v)$ are the new truth, indeterminacy, and falsity values for each vertex v,
- $t'(e), iv'(e), f'(e)$ are the new truth, indeterminacy, and falsity values for each edge e, and the constraints $t' + iv' + f' \leq 3$ are satisfied for both vertices and edges.

□

3.3 General Turiyam Neutrosophic Graph

Next, we consider the General Turiyam Neutrosophic Graph. Similar to how a General Fuzzy Graph is a generalization of a Fuzzy Graph, a General Turiyam Neutrosophic Graph is a generalization of a Turiyam Neutrosophic Graph. The definition is provided below.

Definition 3.10 (General Turiyam Neutrosophic Graph)**.** Let $G = (V, E)$ be a classical graph with vertex set V and edge set E. A *General Turiyam Neutrosophic Graph* $G^T = (V^T, E^T)$ is defined as follows:

- **Turiyam Neutrosophic Vertex Set**: For each vertex $v \in V$, assign four membership degrees:

  $$t(v), \quad iv(v), \quad f(v), \quad lv(v) \in [0, 1],$$

  representing the truth, indeterminacy, falsity, and Turiyam Neutrosophic (liberal) membership degrees of the vertex v, respectively, satisfying:

  $$0 \leq t(v) + iv(v) + f(v) + lv(v) \leq 4.$$

- **Turiyam Neutrosophic Edge Set**: For each edge $e = (u, v) \in E$, assign four membership degrees:

  $$t(e), \quad iv(e), \quad f(e), \quad lv(e) \in [0, 1],$$

  representing the truth, indeterminacy, falsity, and Turiyam Neutrosophic (liberal) membership degrees of the edge e, respectively, satisfying:

  $$0 \leq t(e) + iv(e) + f(e) + lv(e) \leq 4.$$

In a General Turiyam Neutrosophic Graph, the edge membership degrees are independent of the vertex membership degrees. That is, there is no requirement relating $t(e), iv(e), f(e), lv(e)$ to $t(u), iv(u), f(u), lv(u)$ and $t(v), iv(v), f(v), lv(v)$.

Definition 3.11 (Weak General Turiyam Neutrosophic Graph)**.** A *Weak General Turiyam Neutrosophic Graph* $G^T = (V^T, E^T)$ is defined similarly to the General Turiyam Neutrosophic Graph, with the following additional condition:

For each edge $e = (u, v) \in E$, the edge membership degrees satisfy:

$$t(e) \leq t(u) \wedge t(v), \quad iv(e) \leq iv(u) \wedge iv(v), \quad f(e) \geq f(u) \vee f(v), \quad lv(e) \leq lv(u) \wedge lv(v),$$

where $\wedge$ denotes the minimum operation and $\vee$ denotes the maximum operation.

This condition implies that the edge membership degrees are related to the vertex membership degrees, but not necessarily equal to any specific function of them.

Based on the above definition, we will explore the relationships with other graph classes. The following theorem holds.

Theorem 3.12. *Every Turiyam Neutrosophic Graph is a Weak General Turiyam Neutrosophic Graph.*

Proof. In a Turiyam Neutrosophic Graph, the edge membership degrees satisfy the following conditions for each edge $e = (u, v) \in E$:

$$t(e) = t(u) \wedge t(v), \quad iv(e) = iv(u) \wedge iv(v), \quad f(e) = f(u) \vee f(v), \quad lv(e) = lv(u) \wedge lv(v).$$

These conditions are stronger than those required in a Weak General Turiyam Neutrosophic Graph, where the edge membership degrees only need to satisfy:

$$t(e) \leq t(u) \wedge t(v), \quad iv(e) \leq iv(u) \wedge iv(v), \quad f(e) \geq f(u) \vee f(v), \quad lv(e) \leq lv(u) \wedge lv(v).$$

Therefore, every Turiyam Neutrosophic Graph satisfies the properties of a Weak General Turiyam Neutrosophic Graph. □

Theorem 3.13. *Every Weak General Turiyam Neutrosophic Graph is a General Turiyam Neutrosophic Graph.*

Proof. By definition, a Weak General Turiyam Neutrosophic Graph assigns membership degrees to vertices and edges, satisfying the constraints of the General Turiyam Neutrosophic Graph. The additional conditions relating edge and vertex membership degrees in a Weak General Turiyam Neutrosophic Graph do not contradict the General Turiyam Neutrosophic Graph's definition, which allows for edge membership degrees independent of vertex degrees. Therefore, every Weak General Turiyam Neutrosophic Graph is also a General Turiyam Neutrosophic Graph. □

Theorem 3.14. *A Turiyam Neutrosophic Graph is a special case of a General Turiyam Neutrosophic Graph where the edge membership degrees are constrained by the vertex membership degrees.*

Proof. In a Turiyam Neutrosophic Graph, the edge membership degrees are directly related to the vertex membership degrees through specific equalities:

$$t(e) = t(u) \wedge t(v), \quad iv(e) = iv(u) \wedge iv(v), \quad f(e) = f(u) \vee f(v), \quad lv(e) = lv(u) \wedge lv(v).$$

In a General Turiyam Neutrosophic Graph, there are no such constraints; the edge membership degrees are independent of the vertex membership degrees. Therefore, a Turiyam Neutrosophic Graph can be viewed as a General Turiyam Neutrosophic Graph with additional constraints, making it a special case. □

Definition 3.15. Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two General Turiyam Neutrosophic Graphs with underlying crisp graphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$. The union $G = (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is defined as follows:

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u) & \text{if } u \in V_1 - V_2, \\ \sigma_2(u) & \text{if } u \in V_2 - V_1, \\ \max\{\sigma_1(u), \sigma_2(u)\} & \text{if } u \in V_1 \cap V_2. \end{cases}$$

$$(\mu_1 \cup \mu_2)(u, v) = \begin{cases} \mu_1(u, v) & \text{if } (u, v) \in X_1 - X_2, \\ \mu_2(u, v) & \text{if } (u, v) \in X_2 - X_1, \\ \max\{\mu_1(u, v), \mu_2(u, v)\} & \text{if } (u, v) \in X_1 \cap X_2. \end{cases}$$

The c-complement of a General Turiyam Neutrosophic Graph $G = (\sigma, \mu)$ is defined as:

$$\sigma^c = \sigma, \quad \mu^c(u, v) = \begin{cases} 0 & \text{if } \mu(u, v) = 0, \\ 1 - \mu(u, v) & \text{if } \mu(u, v) > 0. \end{cases}$$

The $|\mu|$-complement of a General Turiyam Neutrosophic Graph $G = (\sigma, \mu)$ is defined as:

$$\mu^{|\mu|}(u, v) = \begin{cases} 0 & \text{if } \mu(u, v) = 0, \\ |\sigma(u) \wedge \sigma(v) - \mu(u, v)| & \text{if } \mu(u, v) > 0. \end{cases}$$

Theorem 3.16. *General Turiyam Neutrosophic Graphs are closed under c-complement and $|\mu|$-complement.*

Proof. Based on the definitions of c-complement and $|\mu|$-complement, both operations preserve the conditions that the sum of the truth, indeterminacy, falsity, and liberal values for each vertex and edge does not exceed 4. Therefore, General Turiyam Neutrosophic Graphs remain closed under these operations. □

Theorem 3.17. *The union of two Weak General Turiyam Neutrosophic Graphs is not necessarily a Weak General Turiyam Neutrosophic Graph.*

Proof. Consider two Weak General Turiyam Neutrosophic Graphs $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$. When taking the union $G = (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$, the relationships between edge membership values and vertex membership values may not be preserved, breaking the conditions required for a Weak General Turiyam Neutrosophic Graph. This can occur in cases where the edge values do not correspond to the minimum or maximum of the associated vertex values. □

3.4 Single-Valued Turiyam Neutrosophic Hypergraph

A *Single-Valued Turiyam Neutrosophic Hypergraph* (SVTH) is a generalization of classical hypergraphs and Single-Valued Neutrosophic Hypergraphs, where each vertex and hyperedge is characterized by four degrees: truth, indeterminacy, falsity, and Turiyam Neutrosophic (liberal) state, taking values from the interval $[0, 1]$.

Definition 3.18. A *Single-Valued Turiyam Neutrosophic Hypergraph* $H = (V, E)$ is defined as follows:

- $V = \{v_1, v_2, \ldots, v_n\}$ is a finite set of vertices.
- $E = \{E_1, E_2, \ldots, E_m\}$ is a finite family of Turiyam Neutrosophic subsets of V, where each hyperedge E_i is represented as:
$$E_i = \left\{(v_j, t_{E_i}(v_j), iv_{E_i}(v_j), f_{E_i}(v_j), lv_{E_i}(v_j)) \mid v_j \in V\right\}.$$
For each vertex $v_j \in V$ in a hyperedge E_i, the following four values are assigned:
  - $t_{E_i}(v_j)$ is the truth-membership degree of vertex v_j in hyperedge E_i,
  - $iv_{E_i}(v_j)$ is the indeterminacy-membership degree of vertex v_j in hyperedge E_i,
  - $f_{E_i}(v_j)$ is the falsity-membership degree of vertex v_j in hyperedge E_i,
  - $lv_{E_i}(v_j)$ is the Turiyam Neutrosophic state (liberal value) of vertex v_j in hyperedge E_i.

  The following condition holds for each vertex $v_j \in V$ within a hyperedge E_i:
$$0 \leq t_{E_i}(v_j) + iv_{E_i}(v_j) + f_{E_i}(v_j) + lv_{E_i}(v_j) \leq 4.$$
- **Turiyam Neutrosophic Vertex Set:** The set V represents the vertices of the hypergraph. Each vertex $v_j \in V$ has associated degrees of truth, indeterminacy, falsity, and Turiyam Neutrosophic state within each hyperedge it belongs to.
- **Turiyam Neutrosophic Hyperedge Set:** The set E represents the hyperedges of the hypergraph. Each hyperedge E_i is composed of vertices and their corresponding Turiyam Neutrosophic values: truth $t_{E_i}(v_j)$, indeterminacy $iv_{E_i}(v_j)$, falsity $f_{E_i}(v_j)$, and Turiyam Neutrosophic state $lv_{E_i}(v_j)$.
- **Adjacency of Vertices:** Two vertices u and v are adjacent if they belong to the same hyperedge E_i, i.e., $u, v \in E_i$.
- **Connectivity:** A Single-Valued Turiyam Neutrosophic Hypergraph is connected if every pair of vertices can be connected by a sequence of adjacent vertices.
- **Adjacency of Hyperedges:** Two hyperedges E_i and E_j are adjacent if they share at least one common vertex, i.e., $E_i \cap E_j \neq \emptyset$.
- **Order and Size:**
  - The *order* of the hypergraph is $|V|$, the number of vertices.
  - The *size* of the hypergraph is $|E|$, the number of hyperedges.

- **Cardinality of a Hyperedge:** The degree of a hyperedge E_i is the sum of the truth-membership, indeterminacy-membership, falsity-membership, and Turiyam Neutrosophic state values of the vertices in that hyperedge:

$$d_H(E_i) = \sum_{v_j \in E_i} \left(t_{E_i}(v_j) + iv_{E_i}(v_j) + f_{E_i}(v_j) + lv_{E_i}(v_j) \right).$$

- **Rank and Anti-Rank:**
  - The *rank* of the hypergraph is the maximum cardinality of any hyperedge: $\max_{E_i \in E} d_H(E_i)$.
  - The *anti-rank* of the hypergraph is the minimum cardinality of any hyperedge: $\min_{E_i \in E} d_H(E_i)$.
- **Linear Single-Valued Turiyam Neutrosophic Hypergraph:** A Turiyam Neutrosophic hypergraph is called *linear* if any two distinct vertices belong to at most one common hyperedge, i.e.,

$$|E_i \cap E_j| \leq 1 \quad \text{for all } E_i, E_j \in E.$$

For each vertex v_j in any hyperedge E_i, the following constraint must hold:

$$0 \leq t_{E_i}(v_j) + iv_{E_i}(v_j) + f_{E_i}(v_j) + lv_{E_i}(v_j) \leq 4.$$

This ensures that the sum of the truth, indeterminacy, falsity, and Turiyam Neutrosophic values for any vertex in a hyperedge is valid.

Example 3.19. Consider a Single-Valued Turiyam Neutrosophic Hypergraph $H = (V, E)$ where the vertex set is:

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

and the hyperedge set is:

$$E = \{E_1, E_2, E_3, E_4, E_5, E_6\}$$

The hyperedges are defined as follows:

$$E_1 = \{(v_1, 0.3, 0.4, 0.2, 0.3), (v_3, 0.4, 0.3, 0.2, 0.4)\}$$
$$E_2 = \{(v_1, 0.3, 0.4, 0.2, 0.3), (v_2, 0.5, 0.6, 0.3, 0.2)\}$$
$$E_3 = \{(v_2, 0.5, 0.6, 0.3, 0.2), (v_4, 0.6, 0.4, 0.3, 0.5)\}$$
$$E_4 = \{(v_3, 0.4, 0.3, 0.2, 0.4), (v_6, 0.3, 0.4, 0.2, 0.5)\}$$
$$E_5 = \{(v_3, 0.4, 0.3, 0.2, 0.4), (v_5, 0.6, 0.5, 0.3, 0.2)\}$$
$$E_6 = \{(v_5, 0.6, 0.5, 0.3, 0.2), (v_6, 0.3, 0.4, 0.2, 0.5)\}$$

Each vertex is associated with four membership values: truth (t), indeterminacy (iv), falsity (f), and Turiyam Neutrosophic state (lv). The following conditions for each vertex in every hyperedge are satisfied:

$$0 \leq t(v) + iv(v) + f(v) + lv(v) \leq 4$$

Let's verify this for each vertex:

For hyperedge E_1:

- For v_1, we have $0.3 + 0.4 + 0.2 + 0.3 = 1.2$ (valid).
- For v_3, we have $0.4 + 0.3 + 0.2 + 0.4 = 1.3$ (valid).

For hyperedge E_2:

- For v_1, we have $0.3 + 0.4 + 0.2 + 0.3 = 1.2$ (valid).
- For v_2, we have $0.5 + 0.6 + 0.3 + 0.2 = 1.6$ (valid).

For hyperedge E_3:

- For v_2, we have $0.5 + 0.6 + 0.3 + 0.2 = 1.6$ (valid).
- For v_4, we have $0.6 + 0.4 + 0.3 + 0.5 = 1.8$ (valid).

For hyperedge E_4:

- For v_3, we have $0.4 + 0.3 + 0.2 + 0.4 = 1.3$ (valid).
- For v_6, we have $0.3 + 0.4 + 0.2 + 0.5 = 1.4$ (valid).

For hyperedge E_5:

- For v_3, we have $0.4 + 0.3 + 0.2 + 0.4 = 1.3$ (valid).
- For v_5, we have $0.6 + 0.5 + 0.3 + 0.2 = 1.6$ (valid).

For hyperedge E_6:

- For v_5, we have $0.6 + 0.5 + 0.3 + 0.2 = 1.6$ (valid).
- For v_6, we have $0.3 + 0.4 + 0.2 + 0.5 = 1.4$ (valid).

Thus, all conditions for a Single-Valued Turiyam Neutrosophic Hypergraph are satisfied.

Theorem 3.20. *Every Single-Valued Turiyam Neutrosophic Hypergraph (SVTH) can be transformed into a Single-Valued Neutrosophic Hypergraph (SVNH) by merging the Turiyam Neutrosophic state into the truth, indeterminacy, and falsity values, while maintaining the neutrosophic membership constraints.*

Proof. The proof can be conducted in the same manner as for Single-Valued Turiyam Neutrosophic graphs. □

Corollary 3.21. *Every Linear Single-Valued Turiyam Neutrosophic Hypergraph (SVTH) can be transformed into a Linear Single-Valued Neutrosophic Hypergraph (SVNH) by merging the Turiyam Neutrosophic state into the truth, indeterminacy, and falsity values, while maintaining the neutrosophic membership constraints.*

Proof. The proof can be conducted in the same manner as for Single-Valued Turiyam Neutrosophic graphs. □

3.5 Ambiguous Graph

The concept of Ambiguous sets, as outlined below, has been a subject of research in recent years, with several papers exploring this topic [408, 413, 414]. We now define an Ambiguous Graph by extending this concept to graphs.

It is worth noting that both Ambiguous Graphs and Turiyam Neutrosophic Graphs can be generalized within the framework of Quadripartitioned Neutrosophic Graphs (cf. [417]).

Definition 3.22 (Ambiguous set)**.** [408, 413, 414] Let $H = \{h\}$ be a fixed finite non-empty set of any event h. An AS X for $h \in H$ is expressed as:

$$X = \{\langle h, t_X(h), f_X(h), ta_X(h), fa_X(h)\rangle \mid h \in H\},$$

where t_X, f_X, ta_X, and fa_X denote the true membership function, false membership function, true-ambiguous membership function, and false-ambiguous membership function, respectively.

Here,

$$t_X(h) : H \to]-0, 1[, \quad f_X(h) : H \to]-0, 1[, \quad ta_X(h) : H \to]-0, 1[, \quad \text{and} \quad fa_X(h) : H \to]-0, 1[$$

are called the True Membership Degree (TMD), False Membership Degree (FMD), True-Ambiguous Membership Degree (TAMD), and False-Ambiguous Membership Degree (FAMD) of $h \in H$ in X, respectively.

These membership functions must satisfy the condition:

$$-0 \leq t_X(h) + f_X(h) + ta_X(h) + fa_X(h) \leq 2^+$$

for all $h \in H$. Here, t_X, f_X, ta_X, and fa_X are real standard values or non-standard subsets of $]-0, 1[$.

Definition 3.23 (Ambiguous Graph)**.** Let $G = (V, E)$ be a classical graph with a finite set of vertices $V = \{v_i : i = 1, 2, \ldots, n\}$ and edges $E \subseteq V \times V$. An **Ambiguous Graph** $G^A = (V^A, E^A)$ is defined as follows:

1. **Ambiguous Vertex Set**: For each vertex $v_i \in V$, the ambiguous graph assigns the following membership functions:
$$t_V(v_i), \quad f_V(v_i), \quad ta_V(v_i), \quad fa_V(v_i) : V \to]-0, 1[,$$
where:

- $t_V(v_i)$ is the true membership degree (TMD) of vertex v_i,
- $f_V(v_i)$ is the false membership degree (FMD) of v_i,
- $ta_V(v_i)$ is the true-ambiguous membership degree (TAMD) of v_i,
- $fa_V(v_i)$ is the false-ambiguous membership degree (FAMD) of v_i,

for all $v_i \in V$, satisfying the condition:

$$-0 \leq t_V(v_i) + f_V(v_i) + ta_V(v_i) + fa_V(v_i) \leq 2^+.$$

2. **Ambiguous Edge Set**: For each edge $e_{ij} = (v_i, v_j) \in E$, the ambiguous graph assigns the following membership functions:

$$t_E(e_{ij}), \quad f_E(e_{ij}), \quad ta_E(e_{ij}), \quad fa_E(e_{ij}) : E \rightarrow]-0, 1[,$$

where:

- $t_E(e_{ij})$ is the true membership degree of edge e_{ij},
- $f_E(e_{ij})$ is the false membership degree of e_{ij},
- $ta_E(e_{ij})$ is the true-ambiguous membership degree of e_{ij},
- $fa_E(e_{ij})$ is the false-ambiguous membership degree of e_{ij},

for all $e_{ij} \in E$, satisfying the condition:

$$-0 \leq t_E(e_{ij}) + f_E(e_{ij}) + ta_E(e_{ij}) + fa_E(e_{ij}) \leq 2^+.$$

Here, V^A represents the ambiguous vertex set of the graph G^A, and E^A represents the ambiguous edge set of G^A.

Based on the above definitions, the properties of the Ambiguous Graph are outlined below.

Theorem 3.24. *Every Ambiguous Graph can be transformed into an Intuitionistic Fuzzy Graph by restricting the true-ambiguous membership degree and false-ambiguous membership degree to zero for all vertices and edges.*

Proof. Let $G^A = (V^A, E^A)$ be an Ambiguous Graph as defined. For each vertex $v_i \in V$ and edge $e_{ij} \in E$, the ambiguous graph assigns the membership degrees:

$$\begin{cases} t_V(v_i), & f_V(v_i), & ta_V(v_i), & fa_V(v_i) \\ t_E(e_{ij}), & f_E(e_{ij}), & ta_E(e_{ij}), & fa_E(e_{ij}) \end{cases}$$

satisfying:

$$-0 \leq t_V(v_i) + f_V(v_i) + ta_V(v_i) + fa_V(v_i) \leq 2^+$$

and

$$-0 \leq t_E(e_{ij}) + f_E(e_{ij}) + ta_E(e_{ij}) + fa_E(e_{ij}) \leq 2^+$$

To transform G^A into an Intuitionistic Fuzzy Graph $G^{IF} = (A, B)$, we restrict the true-ambiguous and false-ambiguous membership degrees to zero:

$$ta_V(v_i) = fa_V(v_i) = 0, \quad \forall v_i \in V$$

$$ta_E(e_{ij}) = fa_E(e_{ij}) = 0, \quad \forall e_{ij} \in E$$

Then, define the membership and non-membership functions for G^{IF} as:

$$\mu_A(v_i) = t_V(v_i), \quad \nu_A(v_i) = f_V(v_i)$$

$$\mu_B(e_{ij}) = t_E(e_{ij}), \quad \nu_B(e_{ij}) = f_E(e_{ij})$$

Since $t_V(v_i), f_V(v_i) \in [0, 1]$ and satisfy:

$$-0 \leq t_V(v_i) + f_V(v_i) \leq 2^+$$

which simplifies to:

$$0 \leq \mu_A(v_i) + v_A(v_i) \leq 1$$

because $t_V(v_i) + f_V(v_i) \leq 1$ in standard intervals. Similarly for edges:

$$0 \leq \mu_B(e_{ij}) + v_B(e_{ij}) \leq 1$$

These conditions match those of an Intuitionistic Fuzzy Graph. Therefore, by setting $ta_V(v_i) = fa_V(v_i) = 0$ and $ta_E(e_{ij}) = fa_E(e_{ij}) = 0$, every Ambiguous Graph G^A can be transformed into an Intuitionistic Fuzzy Graph G^{IF}. □

Theorem 3.25. *If in an Ambiguous Graph G^A, the sum of the true membership degree and the true-ambiguous membership degree equals 1 for all vertices and edges, then the graph represents maximum truthfulness under ambiguity.*

Proof. For any vertex $v_i \in V$, suppose:

$$t_V(v_i) + ta_V(v_i) = 1$$

Since all membership degrees are in $[0, 1]$, and the sum of all degrees does not exceed 2, it follows that:

$$f_V(v_i) + fa_V(v_i) = t_V(v_i) + f_V(v_i) + ta_V(v_i) + fa_V(v_i) - (t_V(v_i) + ta_V(v_i)) \leq 2 - 1 = 1$$

Thus, the total falsity (including ambiguity) does not exceed 1, indicating that the vertex has maximum truthfulness under the given ambiguity.

Similarly for edges $e_{ij} \in E$.

This proposition highlights a scenario where the graph exhibits the highest possible degree of truthfulness within the ambiguous framework. □

3.6 Extended Turiyam Neutrosophic Graph

The Extended Turiyam Neutrosophic Graph, which is an extension of the Turiyam Neutrosophic Graph, is described below. This graph handles five types of uncertainties. The definition is provided as follows. Note that a Extended Turiyam Neutrosophic Graph can be generalized within the framework of a pentapartitioned Neutrosophic Graph (cf. [140]).

Definition 3.26. Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices and $E \subseteq V \times V$ is the set of edges.

An *Extended Turiyam Neutrosophic Graph* $G^E = (V^E, E^E)$ is defined as follows:

- **Extended Turiyam Neutrosophic Vertex Set** V^E:
  For each vertex $v \in V$, the following mappings are assigned:

  $$t(v),\ iv(v),\ fv(v),\ lv(v),\ hv(v) : V \to [0, 1],$$

  where:

  - $t(v)$: Truth-membership degree of vertex v,
  - $iv(v)$: Indeterminacy-membership degree of vertex v,
  - $fv(v)$: Falsity-membership degree of vertex v,
  - $lv(v)$: Turiyam Neutrosophic state (liberal value) of vertex v,
  - $hv(v)$: Hesitancy-membership degree of vertex v.

  These degrees represent the five uncertainties associated with each vertex. The following condition holds for each vertex $v \in V$:

  $$0 \leq t(v) + iv(v) + fv(v) + lv(v) + hv(v) \leq 5.$$

- **Extended Turiyam Neutrosophic Edge Set** E^E:
  For each edge $e = (u, v) \in E$, the following mappings are assigned:

  $$t(e),\ iv(e),\ fv(e),\ lv(e),\ hv(e) : E \to [0, 1],$$

  where each degree corresponds to the edge e similarly as for vertices. The following condition holds for each edge $e \in E$:

  $$0 \leq t(e) + iv(e) + fv(e) + lv(e) + hv(e) \leq 5.$$

Theorem 3.27. *Every Extended Turiyam Neutrosophic Graph is a generalization of the Turiyam Neutrosophic Graph.*

Proof. In the Extended Turiyam Neutrosophic Graph $G^E = (V^E, E^E)$, each vertex $v \in V$ has five associated membership degrees: $t(v), iv(v), fv(v), lv(v), hv(v)$.

To show that G^E is a generalization of the Turiyam Neutrosophic Graph G^T, consider the special case where the hesitancy-membership degree $hv(v)$ is zero for all vertices and edges. That is, set $hv(v) = 0$ for all $v \in V$ and $hv(e) = 0$ for all $e \in E$.

With $hv(v) = 0$, the condition for vertices in G^E reduces to:

$$0 \leq t(v) + iv(v) + fv(v) + lv(v) \leq 4,$$

which is exactly the condition in the original Turiyam Neutrosophic Graph definition.

Therefore, the Extended Turiyam Neutrosophic Graph includes the Turiyam Neutrosophic Graph as a special case, making it a generalization. □

Theorem 3.28. *Every Extended Turiyam Neutrosophic Graph can be transformed into a Turiyam Neutrosophic Graph by aggregating the hesitancy-membership degree into the other uncertainties.*

Proof. Given an Extended Turiyam Neutrosophic Graph $G^E = (V^E, E^E)$, for each vertex $v \in V$, we can redistribute the hesitancy-membership degree $hv(v)$ into the other four uncertainties. Define a transformed Turiyam Neutrosophic Graph G^T where:

$$\begin{aligned} t'(v) &= t(v) + \alpha hv(v), \\ iv'(v) &= iv(v) + \beta hv(v), \\ fv'(v) &= fv(v) + \gamma hv(v), \\ lv'(v) &= lv(v) + \delta hv(v), \end{aligned}$$

with coefficients $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + \beta + \gamma + \delta = 1$.

Similarly for edges.

This ensures that the total of the four uncertainties remains within the original bounds:

$$0 \leq t'(v) + iv'(v) + fv'(v) + lv'(v) \leq 4.$$

Thus, the Extended Turiyam Neutrosophic Graph can be transformed into a Turiyam Neutrosophic Graph by appropriately distributing the hesitancy-membership degree. □

Theorem 3.29. *In an Extended Turiyam Neutrosophic Graph, for any edge $e = (u, v)$, the truth-membership degree of the edge satisfies:*

$$t(e) \leq \min\{t(u), t(v)\}.$$

Proof. By the nature of membership degrees, the edge's truth-membership degree cannot exceed the truth-membership degrees of its incident vertices. Suppose, for contradiction, that $t(e) > \min\{t(u), t(v)\}$.

Without loss of generality, assume $t(u) \leq t(v)$, so $\min\{t(u), t(v)\} = t(u)$.

Then $t(e) > t(u)$, which is inconsistent because an edge cannot be more "true" than its least truthful vertex. Therefore, the inequality must hold:

$$t(e) \leq \min\{t(u), t(v)\}.$$

□

Theorem 3.30. *In an Extended Turiyam Neutrosophic Graph, the total uncertainty associated with any vertex or edge does not exceed 5.*

Proof. This follows directly from the definition of the Extended Turiyam Neutrosophic Graph. For any vertex v or edge e, the sum of the uncertainties satisfies:

$$0 \leq t(x) + iv(x) + fv(x) + lv(x) + hv(x) \leq 5,$$

where $x \in V \cup E$.

Since each membership degree $t(x), iv(x), fv(x), lv(x), hv(x)$ lies in the interval $[0, 1]$, the maximum possible sum is when all degrees are at their maximum value of 1:

$$t(x) = iv(x) = fv(x) = lv(x) = hv(x) = 1 \implies \text{Sum} = 5.$$

Therefore, the total uncertainty does not exceed 5. □

3.7 Refined Graph for uncertainty

In recent years, refined sets have been introduced to handle various types of uncertainty [77, 146, 238, 394, 422]. Refined sets adopt the concept of splitting values. By splitting truth values, they generalize fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets. Based on these sets, we define the Refined Graph as follows.

Definition 3.31 (Refined Fuzzy Graph)**.** A *Refined Fuzzy Graph (RFG)* is an extension of a classical fuzzy graph where each vertex and edge is assigned a refined fuzzy value, represented by a Refined Fuzzy Set (RFS).

Let $G = (V, E)$ be a simple graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. The RFG assigns refined fuzzy sets to both vertices and edges, defined as follows:

- **RFS for Vertices**:
  For each vertex $v \in V$, a Refined Fuzzy Set (RFS) is assigned:
  $$\mu_v = (T_1(v), T_2(v), \ldots, T_p(v)),$$
  where $p \geq 2$ and $T_j(v) \in [0, 1]$ for all $j = 1, 2, \ldots, p$. Here, $T_j(v)$ represents the sub-membership degree of vertex v, and the sum of these values is allowed to be adjusted based on the number of values p:
  $$\sum_{j=1}^{p} T_j(v) \leq p.$$
  This allows for greater flexibility in modeling the membership degrees of vertices.
- **RFS for Edges**:
  For each edge $e = (u, v) \in E$, a Refined Fuzzy Set (RFS) is assigned:
  $$\mu_e = (T_1(e), T_2(e), \ldots, T_p(e)),$$
  where $p \geq 2$ and $T_j(e) \in [0, 1]$ for all $j = 1, 2, \ldots, p$. Here, $T_j(e)$ represents the sub-membership degree of edge e, and the sum of these values is similarly allowed to be adjusted:
  $$\sum_{j=1}^{p} T_j(e) \leq p.$$

Theorem 3.32. *If for every vertex $v \in V$ and edge $e \in E$ in a Refined Fuzzy Graph (RFG), all refined membership degrees $T_j(v)$ and $T_j(e)$ are equal and sum to 1, then the RFG reduces to a classical fuzzy graph.*

Proof. In a classical fuzzy graph, each vertex v and edge e has a single membership degree $\mu(v) \in [0, 1]$ and $\mu(e) \in [0, 1]$.

In the RFG, each vertex v has refined membership degrees:

$$\mu_v = (T_1(v), T_2(v), \ldots, T_p(v)),$$

where $T_j(v) \in [0, 1]$ and $\sum_{j=1}^{p} T_j(v) = 1$.

Assuming all $T_j(v)$ are equal, we have:

$$T_j(v) = \frac{1}{p}, \quad \forall j = 1, 2, \ldots, p.$$

Thus:

$$\sum_{j=1}^{p} T_j(v) = p \times \frac{1}{p} = 1.$$

Define the classical membership degree as:

$$\mu(v) = \sum_{j=1}^{p} T_j(v) = 1.$$

Similarly for edges:

$$\mu(e) = \sum_{j=1}^{p} T_j(e) = 1.$$

Therefore, the RFG reduces to a classical fuzzy graph where each vertex and edge has a membership degree of 1. □

Definition 3.33 (Refined Intuitionistic Fuzzy Graph)**.** A *Refined Intuitionistic Fuzzy Graph (RIFG)* extends a classical intuitionistic fuzzy graph by assigning refined membership and non-membership values to both vertices and edges. Each vertex and edge in the graph is associated with a Refined Intuitionistic Fuzzy Set (RIFS), defined as follows:

- **RIFS for Vertices**:
  For each vertex $v \in V$, the Refined Intuitionistic Fuzzy Set (RIFS) is given by:

$$(\mu_v, \nu_v) = \big((T_1(v), T_2(v), \ldots, T_p(v)); (F_1(v), F_2(v), \ldots, F_s(v))\big),$$

  where $p \geq 1$, $s \geq 1$, and $T_j(v), F_l(v) \in [0,1]$ for all $j = 1, 2, \ldots, p$ and $l = 1, 2, \ldots, s$. The sum of membership values $\sum_{j=1}^{p} T_j(v)$ and non-membership values $\sum_{l=1}^{s} F_l(v)$ is adjusted based on the number of truth and falsity values:

$$\sum_{j=1}^{p} T_j(v) + \sum_{l=1}^{s} F_l(v) \leq p + s.$$

- **RIFS for Edges**:
  For each edge $e = (u, v) \in E$, a Refined Intuitionistic Fuzzy Set (RIFS) is given by:

$$(\mu_e, \nu_e) = \big((T_1(e), T_2(e), \ldots, T_p(e)); (F_1(e), F_2(e), \ldots, F_s(e))\big),$$

  where the same conditions apply as for the vertices, and the sum is similarly adjusted:

$$\sum_{j=1}^{p} T_j(e) + \sum_{l=1}^{s} F_l(e) \leq p + s.$$

Theorem 3.34. *If the sum of the refined membership degrees and non-membership degrees for each vertex and edge in a Refined Intuitionistic Fuzzy Graph (RIFG) is 1, and there is only one refined value for each, then the RIFG reduces to a classical intuitionistic fuzzy graph.*

Proof. In RIFG, for each vertex v, we have:

$$\mu_v = (T_1(v)), \quad \nu_v = (F_1(v)),$$

with $T_1(v), F_1(v) \in [0,1]$ and $T_1(v) + F_1(v) = 1$.

Similarly for edges.

This aligns with the definition of a classical intuitionistic fuzzy graph, where each vertex and edge has a membership degree $\mu(v)$ and a non-membership degree $\nu(v)$ such that $\mu(v) + \nu(v) = 1$. □

Definition 3.35. (cf.[424]) Let $G = (V, E)$ be a simple graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Refined Neutrosophic Graph* assigns neutrosophic values to both vertices and edges, where each vertex and each edge has:

- A membership degree split into r values $\mu_1, \mu_2, \ldots, \mu_r$,
- An indeterminacy degree split into s values $\sigma_1, \sigma_2, \ldots, \sigma_s$,
- A non-membership degree split into t values $\nu_1, \nu_2, \ldots, \nu_t$,

such that for each vertex or edge x, the following condition holds:

$$0 \leq \sum_{i=1}^{r} \mu_i(x) + \sum_{i=1}^{s} \sigma_i(x) + \sum_{i=1}^{t} \nu_i(x) \leq n,$$

where $n = r + s + t$.

For each vertex $v \in V$, a neutrosophic n-valued refined set (μ_v, σ_v, ν_v) is assigned, where

$$\mu_v = (\mu_1(v), \ldots, \mu_r(v)), \quad \sigma_v = (\sigma_1(v), \ldots, \sigma_s(v)), \quad \nu_v = (\nu_1(v), \ldots, \nu_t(v)).$$

Similarly, for each edge $e = (u, v) \in E$, a neutrosophic n-valued refined set (μ_e, σ_e, ν_e) is assigned, where

$$\mu_e = (\mu_1(e), \ldots, \mu_r(e)), \quad \sigma_e = (\sigma_1(e), \ldots, \sigma_s(e)), \quad \nu_e = (\nu_1(e), \ldots, \nu_t(e)).$$

Theorem 3.36. *In a Refined Neutrosophic Graph, the total of the refined membership, indeterminacy, and non-membership degrees for any vertex or edge does not exceed $n = r + s + t$.*

Proof. By definition, for each vertex or edge x:

$$0 \leq \sum_{i=1}^{r} \mu_i(x) + \sum_{i=1}^{s} \sigma_i(x) + \sum_{i=1}^{t} \nu_i(x) \leq n.$$

This ensures that the combined degrees do not exceed the total number of refined components n. □

Theorem 3.37. *If $r = s = t = 1$ in a Refined Neutrosophic Graph, it reduces to a classical neutrosophic graph.*

Proof. With $r = s = t = 1$, each vertex and edge has single degrees:

$$\mu(x) = \mu_1(x), \quad \sigma(x) = \sigma_1(x), \quad \nu(x) = \nu_1(x),$$

and the condition becomes:

$$0 \leq \mu(x) + \sigma(x) + \nu(x) \leq 3.$$

After normalizing by dividing each degree by 3, we obtain degrees in $[0, 1]$, matching the classical neutrosophic graph definition. □

Definition 3.38 (Refined Turiyam Neutrosophic Graph)**.** A *Refined Turiyam Neutrosophic Graph* is an extension of a Turiyam Neutrosophic Graph where the truth value, indeterminacy value, falsity value, and liberal value of each vertex and edge are refined into sub-values.

Let $G = (V, E)$ be a classical graph with a finite set of vertices $V = \{v_i : i = 1, 2, \ldots, n\}$ and edges $E = \{(v_i, v_j) : i, j = 1, 2, \ldots, n\}$. The *Refined Turiyam Neutrosophic Graph*, denoted $G^T = (V^T, E^T)$, is defined as follows:

For each vertex $v_i \in V$, the Refined Turiyam Neutrosophic Graph assigns the following mappings:

$$t(v_i) = (\mu_1^t(v_i), \ldots, \mu_r^t(v_i)), \quad iv(v_i) = (\sigma_1^i(v_i), \ldots, \sigma_s^i(v_i)),$$
$$fv(v_i) = (\nu_1^f(v_i), \ldots, \nu_t^f(v_i)), \quad lv(v_i) = (\lambda_1^l(v_i), \ldots, \lambda_q^l(v_i)), \quad (3.1)$$

where:

- $t(v_i)$ is the refined truth value of the vertex v_i, split into r sub-membership values $\mu_1^t(v_i), \ldots, \mu_r^t(v_i)$,
- $iv(v_i)$ is the refined indeterminacy value, split into s values $\sigma_1^i(v_i), \ldots, \sigma_s^i(v_i)$,
- $fv(v_i)$ is the refined falsity value, split into t sub-non-membership values $\nu_1^f(v_i), \ldots, \nu_t^f(v_i)$,
- $lv(v_i)$ is the refined liberal value, split into q values $\lambda_1^l(v_i), \ldots, \lambda_q^l(v_i)$.

These values satisfy the following condition for each vertex v_i:

$$0 \leq \sum_{j=1}^{r} \mu_j^t(v_i) + \sum_{j=1}^{s} \sigma_j^i(v_i) + \sum_{j=1}^{t} \nu_j^f(v_i) + \sum_{j=1}^{q} \lambda_j^l(v_i) \leq n,$$

where $n = r + s + t + q$ is the total number of refined values.

For each edge $(v_i, v_j) \in E$, the Refined Turiyam Neutrosophic Graph assigns the following mappings:

$$t(v_i, v_j) = (\mu_1^t(v_i, v_j), \ldots, \mu_r^t(v_i, v_j)), \quad iv(v_i, v_j) = (\sigma_1^i(v_i, v_j), \ldots, \sigma_s^i(v_i, v_j))$$
$$fv(v_i, v_j) = (\nu_1^f(v_i, v_j), \ldots, \nu_t^f(v_i, v_j)), \quad lv(v_i, v_j) = (\lambda_1^l(v_i, v_j), \ldots, \lambda_q^l(v_i, v_j)),$$

where:

- $t(v_i, v_j)$ is the refined truth value of the edge (v_i, v_j),
- $iv(v_i, v_j)$ is the refined indeterminacy value of the edge,
- $fv(v_i, v_j)$ is the refined falsity value of the edge,
- $lv(v_i, v_j)$ is the refined liberal value of the edge.

These values satisfy the following condition for each edge (v_i, v_j):

$$0 \leq \sum_{j=1}^{r} \mu_j^t(v_i, v_j) + \sum_{j=1}^{s} \sigma_j^i(v_i, v_j) + \sum_{j=1}^{t} \nu_j^f(v_i, v_j) + \sum_{j=1}^{q} \lambda_j^l(v_i, v_j) \leq n.$$

Theorem 3.39. *In a Refined Turiyam Neutrosophic Graph, the sum of the refined truth, indeterminacy, falsity, and liberal degrees for any vertex or edge does not exceed $n = r + s + t + q$.*

Proof. By definition, for each vertex or edge x:

$$0 \leq \sum_{j=1}^{r} \mu_j^t(x) + \sum_{j=1}^{s} \sigma_j^i(x) + \sum_{j=1}^{t} \nu_j^f(x) + \sum_{j=1}^{q} \lambda_j^l(x) \leq n.$$

This ensures the total degrees are within the allowable range. □

Theorem 3.40. *If $r = s = t = q = 1$ in a Refined Turiyam Neutrosophic Graph, it reduces to a classical Turiyam Neutrosophic Graph.*

Proof. Each vertex and edge then has single degrees:

$$t(x) = \mu_1^t(x), \quad iv(x) = \sigma_1^i(x), \quad fv(x) = \nu_1^f(x), \quad lv(x) = \lambda_1^l(x),$$

with the condition:

$$0 \leq t(x) + iv(x) + fv(x) + lv(x) \leq 4.$$

After normalization, this aligns with the classical Turiyam Neutrosophic Graph. □

Definition 3.41 (Refined Extended Turiyam Neutrosophic Graph)**.** A *Refined Extended Turiyam Neutrosophic Graph* is an extension of the Extended Turiyam Neutrosophic Graph that refines the truth, indeterminacy, falsity, liberal, and hesitancy values of both vertices and edges into multiple sub-values. This refined graph allows for more detailed and precise modeling of various types of uncertainties.

Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices and $E \subseteq V \times V$ is the set of edges. The *Refined Extended Turiyam Neutrosophic Graph* $G^R = (V^R, E^R)$ is defined as follows:

For each vertex $v \in V$, the following mappings are assigned:

$$t(v) = (\mu_1^t(v), \ldots, \mu_r^t(v)), \quad iv(v) = (\sigma_1^i(v), \ldots, \sigma_s^i(v)), \quad fv(v) = (\nu_1^f(v), \ldots, \nu_t^f(v)),$$

$$lv(v) = (\lambda_1^l(v), \ldots, \lambda_q^l(v)), \quad hv(v) = (\eta_1^h(v), \ldots, \eta_p^h(v)),$$

where:

- $t(v)$ is the refined truth-membership degree of vertex v, split into r sub-values $\mu_1^t(v), \ldots, \mu_r^t(v)$,
- $iv(v)$ is the refined indeterminacy-membership degree, split into s sub-values $\sigma_1^i(v), \ldots, \sigma_s^i(v)$,
- $fv(v)$ is the refined falsity-membership degree, split into t sub-values $\nu_1^f(v), \ldots, \nu_t^f(v)$,
- $lv(v)$ is the refined liberal value, split into q sub-values $\lambda_1^l(v), \ldots, \lambda_q^l(v)$,
- $hv(v)$ is the refined hesitancy-membership degree, split into p sub-values $\eta_1^h(v), \ldots, \eta_p^h(v)$.

These values represent the five refined uncertainties associated with each vertex, and they satisfy the following condition:

$$0 \leq \sum_{j=1}^{r} \mu_j^t(v) + \sum_{j=1}^{s} \sigma_j^i(v) + \sum_{j=1}^{t} \nu_j^f(v) + \sum_{j=1}^{q} \lambda_j^l(v) + \sum_{j=1}^{p} \eta_j^h(v) \leq n,$$

where $n = r + s + t + q + p$ is the total number of sub-values associated with each vertex.

For each edge $e = (u, v) \in E$, the following mappings are assigned:

$$t(e) = (\mu_1^t(e), \ldots, \mu_r^t(e)), \quad iv(e) = (\sigma_1^i(e), \ldots, \sigma_s^i(e)), \quad fv(e) = (\nu_1^f(e), \ldots, \nu_t^f(e)),$$

$$lv(e) = (\lambda_1^l(e), \ldots, \lambda_q^l(e)), \quad hv(e) = (\eta_1^h(e), \ldots, \eta_p^h(e)),$$

where:

- $t(e)$ is the refined truth-membership degree of edge e,
- $iv(e)$ is the refined indeterminacy-membership degree of edge e,
- $fv(e)$ is the refined falsity-membership degree of edge e,
- $lv(e)$ is the refined liberal value of edge e,
- $hv(e)$ is the refined hesitancy-membership degree of edge e.

These values satisfy the following condition for each edge $e \in E$:

$$0 \leq \sum_{j=1}^{r} \mu_j^t(e) + \sum_{j=1}^{s} \sigma_j^i(e) + \sum_{j=1}^{t} \nu_j^f(e) + \sum_{j=1}^{q} \lambda_j^l(e) + \sum_{j=1}^{p} \eta_j^h(e) \leq n,$$

where $n = r + s + t + q + p$ is the total number of sub-values associated with each edge.

Theorem 3.42. *If $r = s = t = q = p = 1$ in a Refined Extended Turiyam Neutrosophic Graph, it reduces to a classical Extended Turiyam Neutrosophic Graph.*

Proof. With single degrees:

$$t(x) = \mu_1^t(x), \quad iv(x) = \sigma_1^i(x), \quad fv(x) = \nu_1^f(x), \quad lv(x) = \lambda_1^l(x), \quad hv(x) = \eta_1^h(x),$$

and the condition:

$$0 \leq t(x) + iv(x) + fv(x) + lv(x) + hv(x) \leq 5.$$

This matches the classical definition after appropriate normalization. □

Hereafter, this paper will consider the restricted version of the refined graph. As a concrete example, in the case of a Refined Extended Turiyam Neutrosophic Graph, we assume $r = s = t = q = p = c$ (where c is a given constant).

4. Result:Property of Plithogenic graphs

In this section, we explore the properties of Plithogenic graphs. Specifically, we define Plithogenic graphs and General Plithogenic graphs, and then examine their relationships.

4.1 Plithogenic graphs

Recently, Plithogenic Graphs have been proposed as a generalization of Fuzzy Graphs and Turiyam Neutrosophic Graphs, as well as a graphical representation of Plithogenic Sets [426]. Plithogenic Graphs have been developed and are currently being actively studied [241, 412, 449, 451] The definition is provided below.

Definition 4.1. [425, 426] Let S be a universal set, and $P \subseteq S$. A **Plithogenic Set** PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0, 1]^s$ is the **Degree of Appurtenance Function (DAF)**.
- $pCF : Pv \times Pv \to [0, 1]^t$ is the **Degree of Contradiction Function (DCF)**.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. **Reflexivity of Contradiction Function**:

$$pCF(a, a) = 0$$

2. **Symmetry of Contradiction Function**:

$$pCF(a, b) = pCF(b, a)$$

Example 4.2. Here, $s, t \in \{1, 2, 3, 4, 5\}$.

- When $s = t = 1$, PS is called a **Plithogenic Fuzzy Set** and is denoted by PFS.
- When $s = 2, t = 1$, PS is called a **Plithogenic Intuitionistic Fuzzy Set** and is denoted by $PIFS$.
- When $s = 3, t = 1$, PS is called a **Plithogenic Neutrosophic Set** and is denoted by PNS.
- When $s = 4, t = 1$, PS is called a **Plithogenic Turiyam Neutrosophic Set** and is denoted by $PTuS$.
- When $s = 5, t = 1$, PS is called a **Plithogenic Extended Turiyam Neutrosophic Set** and is denoted by $PTuS$.

Definition 4.3. [451] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A **Plithogenic Graph** PG is defined as:

$$PG = (PM, PN)$$

where:

1. **Plithogenic Vertex Set** $PM = (M, l, Ml, adf, aCf)$:
   - $M \subseteq V$ is the set of vertices.
   - l is an attribute associated with the vertices.
   - Ml is the range of possible attribute values.
   - $adf : M \times Ml \to [0, 1]^s$ is the **Degree of Appurtenance Function (DAF)** for vertices.
   - $aCf : Ml \times Ml \to [0, 1]^t$ is the **Degree of Contradiction Function (DCF)** for vertices.
2. **Plithogenic Edge Set** $PN = (N, m, Nm, bdf, bCf)$:
   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the range of possible attribute values.
   - $bdf : N \times Nm \to [0, 1]^s$ is the **Degree of Appurtenance Function (DAF)** for edges.
   - $bCf : Nm \times Nm \to [0, 1]^t$ is the **Degree of Contradiction Function (DCF)** for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. **Edge Appurtenance Constraint**: For all $(x, a), (y, b) \in M \times Ml$:

$$bdf\left((xy), (a, b)\right) \leq \min\{adf(x, a), adf(y, b)\}$$

where $xy \in N$ is an edge between vertices x and y, and $(a, b) \in Nm \times Nm$ are the corresponding attribute values.

2. **Contradiction Function Constraint**: For all $(a, b), (c, d) \in Nm \times Nm$:

$$bCf\left((a, b), (c, d)\right) \leq \min\{aCf(a, c), aCf(b, d)\}$$

3. **Reflexivity and Symmetry of Contradiction Functions**:

$$\begin{aligned} aCf(a, a) &= 0, && \forall a \in Ml \\ aCf(a, b) &= aCf(b, a), && \forall a, b \in Ml \\ bCf(a, a) &= 0, && \forall a \in Nm \\ bCf(a, b) &= bCf(b, a), && \forall a, b \in Nm \end{aligned}$$

Example 4.4.
- When $s = t = 1$, PG is called a **Plithogenic Fuzzy Graph**.
- When $s = 2, t = 1$, PG is called a **Plithogenic Intuitionistic Fuzzy Graph**.
- When $s = 3, t = 1$, PG is called a **Plithogenic Neutrosophic Graph**.

- When $s = 4, t = 1$, PG is called a **Plithogenic Turiyam Neutrosophic Graph**.
- When $s = 5, t = 1$, PG is called a **Plithogenic Extended Turiyam Neutrosophic Graph**.

Theorem 4.5. *Plithogenic Turiyam Neutrosophic Graph is Turiyam Neutrosophic Graph.*

Proof. In a Plithogenic Graph, each vertex and edge can have multiple attributes. By restricting the attributes in the Plithogenic Graph to the four attributes that define a Turiyam Neutrosophic Graph—truth, indeterminacy, falsity, and liberal state—and setting the *Degree of Contradiction Function (DCF)* to zero for all attributes (i.e., no contradictions between attributes), we obtain the structure of a Turiyam Neutrosophic Graph.

Let $v \in V$ be a vertex in the Plithogenic Turiyam Neutrosophic Graph, and $Ml = \{t(v), iv(v), fv(v), lv(v)\}$ be the set of attributes for the vertex. The *Degree of Appurtenance Function (DAF)*, adf, assigns values between 0 and 1 to each attribute. By setting the *Degree of Contradiction Function (DCF)*, aCf, to zero, we eliminate contradictions between these attributes, which is equivalent to the condition in the Turiyam Neutrosophic Graph:

$$0 \leq t(v) + iv(v) + fv(v) + lv(v) \leq 4.$$

Let $(v_i, v_j) \in E$ be an edge in the Plithogenic Turiyam Neutrosophic Graph, and $Nm = \{t(v_i, v_j), iv(v_i, v_j), fv(v_i, v$ be the set of attributes for the edge. The *Degree of Appurtenance Function (DAF)*, bdf, assigns values between 0 and 1 to each edge attribute. By setting the *Degree of Contradiction Function (DCF)*, bCf, to zero, we eliminate contradictions between edge attributes, ensuring that the sum of the edge membership degrees does not exceed 4:

$$0 \leq t(v_i, v_j) + iv(v_i, v_j) + fv(v_i, v_j) + lv(v_i, v_j) \leq 4.$$

By restricting the attributes in a Plithogenic Turiyam Neutrosophic Graph to those of a Turiyam Neutrosophic Graph (truth, indeterminacy, falsity, and liberal state) and setting the Degree of Contradiction Function (DCF) to zero, the Plithogenic Turiyam Neutrosophic Graph reduces to a Turiyam Neutrosophic Graph.

Thus, Plithogenic Turiyam Neutrosophic Graph is a generalization of a Turiyam Neutrosophic Graph. □

Theorem 4.6. *Plithogenic Graphs generalize Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Extended Turiyam Neutrosophic Graphs.*

Proof. As per the definitions:

- When $s = t = 1$, PG reduces to a Plithogenic Fuzzy Graph.
- When $s = 2, t = 1$, PG becomes a Plithogenic Intuitionistic Fuzzy Graph.
- When $s = 3, t = 1$, PG becomes a Plithogenic Neutrosophic Graph.
- When $s = 4, t = 1$, PG becomes a Plithogenic Turiyam Neutrosophic Graph.
- When $s = 5, t = 1$, PG becomes a Plithogenic Extended Turiyam Neutrosophic Graph.

Each of these specialized graphs is a particular case of the Plithogenic Graph with specific values of s and t. Therefore, the Plithogenic Graph encompasses all these types as special cases, thereby generalizing them. □

4.2 Other Graph class related to Plithogenic graphs

These Plithogenic graphs are highly manageable, and the following generalizations are also possible. In this subsection, we will explore their relationships with those graph classes.

First, we consider the General Plithogenic Graph, which is a relaxed version of the Plithogenic Graph. Simply put, this definition is designed to allow for modifications to each graph by incorporating conditions that are suitable for each specific graph, enabling their application in a more flexible manner.

Definition 4.7 (General Plithogenic Graph)**.** A **General Plithogenic Graph** G^{GP} is an extension of the classical Plithogenic Graph that allows for a more flexible and independent treatment of vertices and edges. It is defined as follows:

Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices, and $E \subseteq V \times V$ is a set of edges.

A General Plithogenic Graph $G^{GP} = (PM, PN)$ consists of:

1. **General Plithogenic Vertex Set** PM:

$$PM = (M, l, Ml, adf, aCf)$$

where:

- $M \subseteq V$: Set of vertices.
- l: Attribute associated with the vertices.
- Ml: Range of possible attribute values.
- $adf : M \times Ml \to [0,1]^s$: Degree of Appurtenance Function (DAF) for vertices.
- $aCf : Ml \times Ml \to [0,1]^t$: Degree of Contradiction Function (DCF) for vertices.

2. **General Plithogenic Edge Set** PN:

$$PN = (N, m, Nm, bdf, bCf)$$

where:

- $N \subseteq E$: Set of edges.
- m: Attribute associated with the edges.
- Nm: Range of possible attribute values.
- $bdf : N \times Nm \to [0,1]^s$: Degree of Appurtenance Function (DAF) for edges.
- $bCf : Nm \times Nm \to [0,1]^t$: Degree of Contradiction Function (DCF) for edges.

The General Plithogenic Graph G^{GP} only needs to satisfy the following **Reflexivity and Symmetry** properties of the Contradiction Functions:

- Reflexivity and Symmetry of Contradiction Functions:

$$\begin{aligned} aCf(a,a) &= 0, && \forall a \in Ml \\ aCf(a,b) &= aCf(b,a), && \forall a,b \in Ml \\ bCf(a,a) &= 0, && \forall a \in Nm \\ bCf(a,b) &= bCf(b,a), && \forall a,b \in Nm \end{aligned}$$

Theorem 4.8. *Plithogenic Graph is General Plithogenic Graph.*

Proof. Obviously holds. □

We consider transforming this graph into various other graphs. As mentioned earlier, this definition is intended to facilitate modifications to each graph by incorporating conditions tailored to each specific graph, allowing for more flexible application.

4.2.1 Relation for spherical fuzzy graphs

For example, we can refer to spherical fuzzy graphs [37]. In the field of fuzzy graphs, numerous studies have been conducted on spherical fuzzy graphs [121, 199, 209, 245, 301].

Theorem 4.9. *The following are examples of related graph classes for spherical fuzzy graphs, including but not limited to:*

- *T-Spherical Fuzzy Graphs[199]*
- *Spherical Fuzzy Labelling Graphs[122]*
- *spherical fuzzy digraphs[302]*
- *T-spherical fuzzy Hamacher graphs[79]*
- *Regular spherical fuzzy graph[310]*
- *Spherical Fuzzy Cycle graph[274]*
- *Spherical Fuzzy Tree graph[274]*
- *Pseudo regular spherical fuzzy graphs[210]*
- *Spherical neutrosophic graph[13]*

Proof. Refer to each reference as needed. □

The definition is provided below[37].

Definition 4.10. [37] A *spherical fuzzy graph (SFG)* on an underlying set V is a pair $G = (M, N)$, where M is a spherical fuzzy set in V and N is a spherical fuzzy relation on $V \times V$, such that:

$$\alpha_N(a,b) \leq \min\{\alpha_M(a), \alpha_M(b)\},$$

$$\gamma_N(a,b) \leq \min\{\gamma_M(a), \gamma_M(b)\},$$

$$\beta_N(a,b) \leq \max\{\beta_M(a), \beta_M(b)\},$$

where α is known as the degree of truthness, γ is known as the degree of abstinence, and β is known as the degree of falseness. These satisfy the following condition for all $a, b \in V$:

$$0 \leq \alpha_N^2(a,b) + \gamma_N^2(a,b) + \beta_N^2(a,b) \leq 1,$$

where M is the spherical fuzzy vertex set and N is the spherical fuzzy edge set of G.

Theorem 4.11. *General Plithogenic graphs with $s = 3$ and $t = 1$ can be transformed into spherical fuzzy graphs.*

Proof. We want to show that a General Plithogenic Graph G^{GP} with parameters $s = 3$ and $t = 1$ can be transformed into a spherical fuzzy graph (SFG).

By definition, the General Plithogenic Graph $G^{GP} = (PM, PN)$ is composed of a vertex set PM and an edge set PN. These sets contain Degree of Appurtenance Functions (DAF) and Degree of Contradiction Functions (DCF) for vertices and edges:

For the vertex set PM:

$$adf : M \times Ml \to [0,1]^s$$

Since $s = 3$, the DAF for vertices will have three components for each vertex $v \in M$, say $adf(v) = (\mu(v), \eta(v), \nu(v))$, representing membership, abstinence, and non-membership degrees, respectively.

For the edge set PN:

$$bdf : N \times Nm \to [0,1]^s$$

Similarly, the DAF for edges will have three components $bdf(e) = (\mu(e), \eta(e), \nu(e))$, representing membership, abstinence, and non-membership degrees for each edge $e \in N$.

The DCF aCf has $t = 1$, meaning it is a single-valued function measuring contradiction between attributes. This will not affect the transformation into a spherical fuzzy graph since a spherical fuzzy graph's main components involve membership, abstinence, and non-membership values.

A spherical fuzzy graph $G = (M, N)$ is characterized by the following conditions:

$$\alpha_N(a,b) \leq \min\{\alpha_M(a), \alpha_M(b)\},$$

$$\gamma_N(a,b) \leq \min\{\gamma_M(a), \gamma_M(b)\},$$

$$\beta_N(a,b) \leq \max\{\beta_M(a), \beta_M(b)\},$$

with the constraint

$$0 \leq \alpha_N^2(a,b) + \gamma_N^2(a,b) + \beta_N^2(a,b) \leq 1.$$

Since the General Plithogenic Graph has DAF values for both vertices and edges in the form (μ, η, ν), where μ is the degree of membership, η is the degree of abstinence, and ν is the degree of non-membership, these can directly correspond to α, γ, and β in a spherical fuzzy graph. □

4.2.2 Relation for General graphs

General (Intuitionistic) fuzzy graphs, weak General (Intuitionistic) fuzzy graphs, General Turiyam Neutrosophic graphs, and weak General Turiyam Neutrosophic graphs can all be generalized using Plithogenic graphs. The following properties hold.

Theorem 4.12. *A Plithogenic graph with $s = 1$ and $t = 1$ satisfies all the conditions required to transform it into general fuzzy graph.*

Proof. Obviously holds. □

Theorem 4.13. *A General Plithogenic Graph with $s = 1$ and $t = 1$ satisfies all the conditions required to transform it into weak general fuzzy graph.*

Proof. Obviously holds. □

Theorem 4.14. *A General Plithogenic Graph with $s = 2$ and $t = 1$ satisfies all the conditions required to transform it into general Intuitionistic fuzzy graph.*

Proof. Obviously holds. □

Theorem 4.15. *A General Plithogenic Graph with $s = 2$ and $t = 1$ satisfies all the conditions required to transform it into weak general Intuitionistic fuzzy graph.*

Proof. Obviously holds. □

Theorem 4.16. *A General Plithogenic Graph with $s = 4$ and $t = 1$ satisfies all the conditions required to transform it into general Turiyam Neutrosophic graph.*

Proof. Obviously holds. □

Theorem 4.17. *A General Plithogenic Graph with $s = 4$ and $t = 1$ satisfies all the conditions required to transform it into weak general Turiyam Neutrosophic graph.*

Proof. Obviously holds. □

4.2.3 Relation for Pythagorean fuzzy graphs

Next, we consider the Pythagorean fuzzy graph. Pythagorean fuzzy graphs have also been extensively studied in the field of fuzzy theory [27, 28, 35, 401, 467].

Theorem 4.18. *The following are examples of related graph classes for Pythagorean fuzzy graphs, including but not limited to:*

- *Cubic Pythagorean fuzzy graphs [321]*
- *Complex Pythagorean Dombi fuzzy graphs [29]*
- *Interval-valued Pythagorean fuzzy graphs[36]*
- *Interval-valued complex Pythagorean fuzzy graph[441]*
- *Pythagorean Dombi fuzzy graphs[20]*
- *Pythagorean fuzzy soft graphs[401]*
- *Pythagorean neutrosophic fuzzy graphs[12]*
- *Complex pythagorean fuzzy planar graphs[18]*
- *Pythagorean neutrosophic Dombi fuzzy graphs [120]*
- *Complex Pythagorean fuzzy threshold graphs[16]*
- *Pythagorean Dombi fuzzy soft graphs[42]*
- *Pythagorean fuzzy incidence graphs[44, 405]*

Proof. Refer to each reference as needed. □

The following theorem holds.

Definition 4.19. [35] A *Pythagorean fuzzy graph (PFG)* on a nonempty set V is a pair $G = (P, Q)$, where:

- P is a Pythagorean fuzzy set (PFS) on V,
- Q is a Pythagorean fuzzy relation (PFR) on $V \times V$,

satisfying the following conditions for all $u, v \in V$:

$$Q(uv) \leq P(u) \wedge P(v),$$

$$Q(uv) \geq P(u) \vee P(v),$$

where $Q : V \times V \to [0,1]$ and $\overline{Q} : V \times V \to [0,1]$ represent the membership and non-membership functions of Q, respectively. These functions satisfy the Pythagorean fuzzy condition:

$$0 \leq Q^2(uv) + \overline{Q}^2(uv) \leq 1 \quad \forall uv \in E.$$

Theorem 4.20. *A General Plithogenic Graph with $s = 2$ and $t = 1$, representing membership and non-membership degrees, can be transformed into a Pythagorean fuzzy graph (PFG).*

Proof. To prove that a General Plithogenic Graph with $s = 2$ and $t = 1$ can be transformed into a Pythagorean fuzzy graph (PFG), we start by recalling the definition of a PFG. A PFG on a nonempty set V is a pair $G = (P, Q)$, where:

- P is a Pythagorean fuzzy set (PFS) on V,
- Q is a Pythagorean fuzzy relation (PFR) on $V \times V$.

In a General Plithogenic Graph with $s = 2$, the two uncertainty components μ_1 and μ_2 represent membership and non-membership degrees, respectively. We can map these directly to the PFG structure by defining:

$$Q(uv) = \mu_1(uv) \quad \text{and} \quad \overline{Q}(uv) = \mu_2(uv),$$

for each edge $uv \in E$.

The Pythagorean condition requires that:

$$0 \leq Q^2(uv) + \overline{Q}^2(uv) \leq 1.$$

Since μ_1 and μ_2 in the Plithogenic Graph satisfy this condition by definition (i.e., $\mu_1^2 + \mu_2^2 \leq 1$), they conform to the Pythagorean fuzzy requirements.

Furthermore, the Degree of Appurtenance Function (DAF) in the Plithogenic Graph ensures that:

$$\mu_1(uv) \leq \min(\mu_1(u), \mu_1(v)),$$

which aligns with the requirement that $Q(uv) \leq \min(P(u), P(v))$ in a PFG. □

4.2.4 Relation for Hesitancy Fuzzy Graphs

Next, we consider the Hesitancy Fuzzy Graph. Like other classes of fuzzy graphs, the Hesitancy Fuzzy Graph has been the subject of extensive research [201, 216, 344, 362, 366]. This type of graph introduces the concept of the hesitancy of an element $v_i \in V$ into the fuzzy graph structure.

Theorem 4.21. *The following are examples of related graph classes for Hesitancy fuzzy graph, including but not limited to:*

- *Complex Hesitant Fuzzy Graph [6]*
- *Constant hesitancy fuzzy graphs[343]*
- *Bipolar hesitancy fuzzy graph[217, 340]*
- *Hesitancy fuzzy magic labeling graph[374]*
- *Dual hesitant fuzzy graphs[87]*
- *Regular Hesitancy Fuzzy Soft Graphs[272]*
- *Hesitant fuzzy hypergraphs[194]*

Proof. Refer to each reference as needed. □

The definition is provided below.

Definition 4.22. [344] A Hesitancy Fuzzy Graph is of the form $G = (V, E)$, where

- $V = \{v_1, v_2, v_3, \ldots, v_n\}$ such that $\mu_1 : V \to [0,1]$, $\gamma_1 : V \to [0,1]$, and $\beta_1 : V \to [0,1]$ denote the degree of membership, non-membership, and hesitancy of the element $v_i \in V$, respectively, and
$$\mu_1(v_i) + \gamma_1(v_i) + \beta_1(v_i) = 1$$
for every $v_i \in V$, where
$$\beta_1(v_i) = 1 - [\mu_1(v_i) + \gamma_1(v_i)] \tag{1}$$
- $E \subseteq V \times V$ where $\mu_2 : V \times V \to [0,1]$, $\gamma_2 : V \times V \to [0,1]$, and $\beta_2 : V \times V \to [0,1]$ are such that
$$\mu_2(v_i, v_j) \leq \min[\mu_1(v_i), \mu_1(v_j)] \tag{2}$$
$$\gamma_2(v_i, v_j) \leq \max[\gamma_1(v_i), \gamma_1(v_j)] \tag{3}$$
$$\beta_2(v_i, v_j) \leq \min[\beta_1(v_i), \beta_1(v_j)] \tag{4}$$
and
$$0 \leq \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) + \beta_2(v_i, v_j) \leq 1$$
for every $(v_i, v_j) \in E$.

Theorem 4.23. *A General Plithogenic Graph with $s = 3$ and $t = 1$ can be transformed into a Hesitancy Fuzzy Graph.*

Proof. To prove that a General Plithogenic Graph with $s = 3$ and $t = 1$ can be transformed into a Hesitancy Fuzzy Graph, we start by examining the definitions of both graphs and establishing the required mappings.

A General Plithogenic Graph is defined as:
$$PS = (P, v, Pv, pdf, pCF)$$
where:

- P is a set of vertices.
- v is an attribute, and Pv is the set of possible values for v.
- **Degree of Appurtenance Function (DAF)** $pdf : P \times Pv \to [0,1]^s$, where $s = 3$ implies three components representing uncertainty.
- **Degree of Contradiction Function (DCF)** $pCF : Pv \times Pv \to [0,1]^t$, where $t = 1$.

The three components of the DAF in this case represent:

1. $\mu_1(v)$: Membership degree.
2. $\gamma_1(v)$: Non-membership degree.
3. $\beta_1(v)$: Hesitancy degree, such that $\mu_1(v) + \gamma_1(v) + \beta_1(v) = 1$.

In a Hesitancy Fuzzy Graph $G = (V, E)$, each vertex $v_i \in V$ has associated values $\mu_1(v_i)$, $\gamma_1(v_i)$, and $\beta_1(v_i)$ satisfying the same condition $\mu_1(v_i) + \gamma_1(v_i) + \beta_1(v_i) = 1$. For edges $(v_i, v_j) \in E$, the conditions:
$$\mu_2(v_i, v_j) \leq \min[\mu_1(v_i), \mu_1(v_j)],$$
$$\gamma_2(v_i, v_j) \leq \max[\gamma_1(v_i), \gamma_1(v_j)],$$
$$\beta_2(v_i, v_j) \leq \min[\beta_1(v_i), \beta_1(v_j)]$$
must hold.

To transform a Plithogenic Graph into a Hesitancy Fuzzy Graph:

1. Map the DAF components directly:
$$\mu_1(v) = pdf_1(v), \quad \gamma_1(v) = pdf_2(v), \quad \beta_1(v) = pdf_3(v).$$
2. This satisfies $\mu_1(v) + \gamma_1(v) + \beta_1(v) = 1$ as required by both graph definitions.
3. For edges, map:
$$\mu_2(v_i, v_j) = pdf_1(v_i, v_j), \quad \gamma_2(v_i, v_j) = pdf_2(v_i, v_j), \quad \beta_2(v_i, v_j) = pdf_3(v_i, v_j).$$
These values naturally satisfy the conditions of the Hesitancy Fuzzy Graph due to the properties of DAF and DCF.

Finally, since $t = 1$, the single contradiction value in the Plithogenic Graph ensures consistent relationships, and thus, the transformation is valid. This completes the proof. □

4.2.5 Relation for Intuitionistic Hesitancy Fuzzy Graphs

Next, we consider the Intuitionistic Hesitancy Fuzzy Graph, a newly proposed class of graphs in recent years. It is known to be a generalized class of Hesitancy Fuzzy Graphs. The definition is provided below [319, 320].

Definition 4.24. [320] An Intuitionistic Hesitancy Fuzzy Graph is $G = (V, E, \sigma, \mu)$, where:

- V is the vertex set.
- $\lambda_1, \delta_1, \rho_1 : V \to [0, 1]$ represent the degree of membership (MS), non-membership (NMS), and hesitancy of $v \in V$, respectively, satisfying:

$$0 \leq \lambda_1(v) + \delta_1(v) + \rho_1(v) \leq 1$$

- $\lambda_2, \delta_2, \rho_2 : V \times V \to [0, 1]$ represent the degree of membership (MS), non-membership (NMS), and hesitancy of $x = (u, v) \in V \times V$, respectively, satisfying:

$$\lambda_2(x) \leq \min\{\lambda_1(u), \lambda_1(v)\}$$

$$\delta_2(x) \leq \max\{\delta_1(u), \delta_1(v)\}$$

$$\rho_2(x) \leq \min\{\rho_1(u), \rho_1(v)\}$$

and

$$0 \leq \lambda_2(x) + \delta_2(x) + \rho_2(x) \leq 1, \quad \forall x \in V \times V$$

Theorem 4.25. *A General Plithogenic Graph with $s = 3$ and $t = 1$, representing membership and non-membership degrees, can be transformed into a Intuitionistic Hesitancy Fuzzy Graphs.*

Proof. It can be proven in the same way as the previous theorem. □

4.2.6 Relation for picture fuzzy graph

Next, we consider the picture fuzzy graph. The picture fuzzy graph is a generalization of fuzzy graphs and has been extensively studied, with many applications also being explored [68, 263].

Theorem 4.26. *The following are examples of related graph classes for picture fuzzy graph, including but not limited to:*

- *Picture Dombi Fuzzy Graph [311]*
- *Picture fuzzy line graphs[126]*
- *Picture fuzzy planar graphs[92]*
- *Picture fuzzy tolerance graphs[135]*
- *Picture fuzzy ϕ-tolerance competition graphs[136]*
- *Interval-Valued Picture Fuzzy Graph [368]*
- *Picture Fuzzy Incidence Graph [333]*
- *Picture fuzzy threshold graphs[135]*
- *Picture Fuzzy Soft Graph [123]*
- *Cayley Picture Fuzzy Graph [256]*
- *m-Polar Picture Fuzzy Graphs[257]*
- *Interval-Valued Picture (S, T)-Fuzzy Graph [75]*
- *q-Rung Picture Fuzzy Graph [26]*
- *Mixed Picture Fuzzy Graph [327]*
- *Picture Fuzzy Directed Hypergraphs[255]*

- *Picture fuzzy cubic graphs[258]*
- *Picture Fuzzy Digraph [302]*
- *Balanced Picture Fuzzy Graph [67]*

Proof. Refer to each reference as needed. □

The definition is provided below [320].

Definition 4.27. [320] Let $G^* = (V, E)$ be a graph. A pair $G = (A, B)$ is called a *picture fuzzy graph* on G^*, where:

- $A = (\mu_A, \eta_A, \nu_A)$ is a picture fuzzy set on V,
- $B = (\mu_B, \eta_B, \nu_B)$ is a picture fuzzy set on $E \subseteq V \times V$,

such that for each arc $uv \in E$, the following conditions hold:

$$\mu_B(u, v) \leq \min(\mu_A(u), \mu_A(v)),$$
$$\eta_B(u, v) \leq \min(\eta_A(u), \eta_A(v)),$$
$$\nu_B(u, v) \geq \max(\nu_A(u), \nu_A(v)).$$

Theorem 4.28. *A Plithogenic graph with $s = 2$ and $t = 1$, representing membership and non-membership degrees, can be transformed into a picture fuzzy graph.*

Proof. To prove that a Plithogenic graph with $s = 2$ and $t = 1$ can be transformed into a picture fuzzy graph, we start by recalling the definition of a picture fuzzy graph $G = (A, B)$ on a graph $G^* = (V, E)$. It consists of:

- $A = (\mu_A, \eta_A, \nu_A)$, a picture fuzzy set on V,
- $B = (\mu_B, \eta_B, \nu_B)$, a picture fuzzy set on $E \subseteq V \times V$,

such that for each edge $uv \in E$, the conditions hold:

$$\mu_B(u, v) \leq \min(\mu_A(u), \mu_A(v)),$$
$$\eta_B(u, v) \leq \min(\eta_A(u), \eta_A(v)),$$
$$\nu_B(u, v) \geq \max(\nu_A(u), \nu_A(v)).$$

In a Plithogenic graph with $s = 2$ and $t = 1$, we have two components representing uncertainty:

- μ_1: Membership degree,
- μ_2: Non-membership degree.

To transform this into a picture fuzzy graph, we map these components as follows:

$$\mu_A(u) = \mu_1(u), \quad \eta_A(u) = \mu_2(u), \quad \nu_A(u) = 1 - (\mu_1(u) + \mu_2(u)),$$

for each vertex $u \in V$. Similarly, for edges $uv \in E$, we define:

$$\mu_B(u, v) = \mu_1(uv), \quad \eta_B(u, v) = \mu_2(uv), \quad \nu_B(u, v) = 1 - (\mu_1(uv) + \mu_2(uv)).$$

These definitions ensure that the conditions of the picture fuzzy graph are satisfied because:

$$\mu_B(u, v) \leq \min(\mu_A(u), \mu_A(v)),$$
$$\eta_B(u, v) \leq \min(\eta_A(u), \eta_A(v)),$$
$$\nu_B(u, v) \geq \max(\nu_A(u), \nu_A(v)),$$

naturally follow from the properties of the membership, non-membership, and hesitancy degrees.

Therefore, a Plithogenic graph with $s = 2$ and $t = 1$ can be effectively transformed into a picture fuzzy graph by interpreting its components accordingly. □

4.2.7 Relation for Ambiguous Graph

Next, we consider the relations for the Ambiguous Graph. The following holds:

Theorem 4.29. *A General Plithogenic Graph with $s = 4$ and $t = 1$ can be transformed into an Ambiguous Graph.*

Proof. The proof can be constructed in a manner similar to the previous theorem. □

4.2.8 Relation for Quadripartitioned Neutrosophic Set and Graph

The Quadripartitioned Neutrosophic Set and Graph are recently defined sets and graphs, and several studies have been conducted on them. Related concepts include Quadripartitioned Neutrosophic Soft Graphs [226], Quadripartitioned Neutrosophic Competition Graph [218], Quadripartitioned Single-Valued Neutrosophic Graph [219], Quadripartitioned Neutrosophic Pythagorean Soft Set [364], and Interval Quadripartitioned Neutrosophic Sets [361].

Definition 4.30 (Quadripartitioned Neutrosophic Set (QPNS))**.** (cf.[361]) A **quadripartitioned neutrosophic set** A on a universal set Y is defined as:

$$A = \{(y, t_A(y), c_A(y), u_A(y), f_A(y)) : y \in Y\},$$

where $t_A(y), c_A(y), u_A(y), f_A(y) \in [0, 1]$ represent the truth, contradiction, unknown, and falsity membership functions of each $y \in Y$, respectively. The membership functions $t_A(y), c_A(y), u_A(y), f_A(y)$ must satisfy the condition:

$$0 \leq t_A(y) + c_A(y) + u_A(y) + f_A(y) \leq 4.$$

Definition 4.31 (Quadripartitioned Neutrosophic Graph (QPNG))**.** [225] Let V be a finite set of vertices, and let $E \subseteq \{\{u, v\} : u, v \in V \text{ with } u \neq v\}$ be a set of edges. Let A be a quadripartitioned neutrosophic set (QPNS) on V, and B be a QPNS on E, such that for each $\{u, v\} \in E$, the following conditions hold:

$$t_B(u, v) \leq \min\{t_A(u), t_A(v)\},$$

$$c_B(u, v) \leq \min\{c_A(u), c_A(v)\},$$

$$u_B(u, v) \leq \max\{u_A(u), u_A(v)\},$$

$$f_B(u, v) \leq \max\{f_A(u), f_A(v)\}.$$

Then, $G = (A, B, V, E)$ is called a **quadripartitioned neutrosophic graph (QPNG)**.

1. Each $v \in V$ is called a vertex of G.
2. Each $\{u, v\} \in E$ is called an edge of G.

Theorem 4.32. *A General Plithogenic Graph with $s = 4$ and $t = 1$ can be transformed into an Quadripartitioned Neutrosophic Graph.*

Proof. The proof can be constructed in a manner similar to the previous theorem. □

4.2.9 Relation for Pentapartitioned Neutrosophic Set and Graph

The Pentapartitioned Neutrosophic Set and Graph are recently defined sets and graphs, and several studies have been conducted on them [109, 119, 139, 140, 363].

Definition 4.33 (Pentapartitioned Neutrosophic Set (PPNS) [300])**.** A **pentapartitioned neutrosophic set** A on a universal set Y is defined as:

$$A = \{(y, t_A(y), c_A(y), g_A(y), u_A(y), f_A(y)) : y \in Y\},$$

where $t_A(y), c_A(y), g_A(y), u_A(y), f_A(y) \in [0, 1]$ represent the truth, contradiction, ignorance, unknown, and falsity membership functions of each $y \in Y$, respectively. The membership functions $t_A(y), c_A(y), g_A(y), u_A(y), f_A(y)$ must satisfy the condition:

$$0 \leq t_A(y) + c_A(y) + g_A(y) + u_A(y) + f_A(y) \leq 5.$$

Definition 4.34 (Pentapartitioned Neutrosophic Graph (PPNG) [363])**.** Let V be a finite set of vertices, and let $E \subseteq \{\{u, v\} : u, v \in V \text{ with } u \neq v\}$ be a set of edges. Let A be a pentapartitioned neutrosophic set (PPNS) on V, and B be a PPNS on E, such that for each $\{u, v\} \in E$, the following conditions hold:

$$t_B(u, v) \leq \min\{t_A(u), t_A(v)\},$$

$$c_B(u, v) \geq \max\{c_A(u), c_A(v)\},$$

$$g_B(u, v) \geq \max\{g_A(u), g_A(v)\},$$

$$u_B(u, v) \geq \max\{u_A(u), u_A(v)\},$$

$$f_B(u, v) \geq \max\{f_A(u), f_A(v)\}.$$

Then, $G = (A, B, V, E)$ is called a **pentapartitioned neutrosophic graph (PPNG)**.

1. Each $v \in V$ is called a vertex of G.
2. Each $\{u, v\} \in E$ is called an edge of G.

Theorem 4.35. *A General Plithogenic Graph with $s = 5$ and $t = 1$ can be transformed into an Pentapartitioned Neutrosophic Graph.*

Proof. The proof can be constructed in a manner similar to the previous theorem. □

4.2.10 Relation for Penta-Valued Fuzzy Set and graph

The Penta-Valued Fuzzy Set has been previously proposed [347,349]. The graph-based definition of this concept is provided below.

Definition 4.36 (Penta-Valued Fuzzy Graph)**.** (cf.[347, 349]) Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices, and $E \subseteq V \times V$ is a set of edges. A **penta-valued fuzzy graph** G^{PVF} is defined by associating a penta-valued fuzzy set to both the vertices and the edges of the graph.

1. **Penta-Valued Fuzzy Vertex Set:** For each vertex $v \in V$, the penta-valued fuzzy graph assigns the following membership functions:

   $$\mu_V(v), \quad \lambda_V(v), \quad \pi_V(v), \quad \omega_V(v), \quad \nu_V(v) : V \to [0, 1],$$

   where:

   - $\mu_V(v)$ represents the strong membership degree of the vertex v,
   - $\lambda_V(v)$ represents the weak membership degree of v,
   - $\pi_V(v)$ represents the uncertainty degree of v,
   - $\omega_V(v)$ represents the weak non-membership degree of v,
   - $\nu_V(v)$ represents the strong non-membership degree of v.

   These membership functions must satisfy the condition:

   $$\mu_V(v) + \lambda_V(v) + \pi_V(v) + \omega_V(v) + \nu_V(v) = 1 \quad \forall v \in V.$$

2. **Penta-Valued Fuzzy Edge Set:** For each edge $e = (u, v) \in E$, the penta-valued fuzzy graph assigns the following membership functions:

   $$\mu_E(e), \quad \lambda_E(e), \quad \pi_E(e), \quad \omega_E(e), \quad \nu_E(e) : E \to [0, 1],$$

   where:

   - $\mu_E(e)$ represents the strong membership degree of the edge e,
   - $\lambda_E(e)$ represents the weak membership degree of e,
   - $\pi_E(e)$ represents the uncertainty degree of e,
   - $\omega_E(e)$ represents the weak non-membership degree of e,
   - $\nu_E(e)$ represents the strong non-membership degree of e.

These membership functions must satisfy the condition:

$$\mu_E(e) + \lambda_E(e) + \pi_E(e) + \omega_E(e) + \nu_E(e) = 1 \quad \forall e \in E.$$

The penta-valued fuzzy graph G^{PVF} is represented as:

$$G^{PVF} = (V^{PVF}, E^{PVF}),$$

where:

- $V^{PVF} = \{(v, \mu_V(v), \lambda_V(v), \pi_V(v), \omega_V(v), \nu_V(v)) : v \in V\}$ is the penta-valued fuzzy vertex set.
- $E^{PVF} = \{(e, \mu_E(e), \lambda_E(e), \pi_E(e), \omega_E(e), \nu_E(e)) : e \in E\}$ is the penta-valued fuzzy edge set.

Theorem 4.37. *A General Plithogenic Graph with $s = 5$ and $t = 1$ satisfies all the conditions required to transform it into a Penta-Valued Fuzzy Graph.*

Proof. The proof can be constructed in a manner similar to the previous theorem. □

4.2.11 Relation for Four-Valued Fuzzy Set and graph

Four-Valued Fuzzy Sets and graphs are extensions of Tree-Valued Fuzzy Sets and Fuzzy Graphs. Several studies have been published on this topic [346, 348]. The definition is provided below. It is worth noting that a Four-Valued Fuzzy Set can be generalized within the framework of a Quadripartitioned Neutrosophic Set (cf. [417]).

Definition 4.38 (Four-Valued Fuzzy Set)**.** [346] Let X be a non-empty set (the universe of discourse). A **four-valued fuzzy set** W on X is defined as:

$$W = \{(x, \mu(x), \lambda(x), \omega(x), \nu(x)) : x \in X\},$$

where

$$\mu, \lambda, \omega, \nu : X \to [0, 1]$$

are the membership functions representing the following components:

- $\mu(x)$: True membership function indicating the degree to which x is strongly true.
- $\lambda(x)$: Contingent true membership function indicating the degree to which x is weakly true.
- $\omega(x)$: Contingent false membership function indicating the degree to which x is weakly false.
- $\nu(x)$: False membership function indicating the degree to which x is strongly false.

These membership functions must satisfy the condition:

$$\mu(x) + \lambda(x) + \omega(x) + \nu(x) = 1 \quad \forall x \in X.$$

The overall membership degree $w(x)$ of an element x in W is determined by the weighted sum:

$$w(x) = 1 \cdot \mu(x) + \frac{2}{3} \cdot \lambda(x) + \frac{1}{3} \cdot \omega(x) + 0 \cdot \nu(x).$$

Definition 4.39 (Four-Valued Fuzzy Graph)**.** Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices, and $E \subseteq V \times V$ is a set of edges. A **four-valued fuzzy graph** G^{FVF} is defined by associating a four-valued fuzzy set to both the vertices and the edges of the graph.

For each vertex $v \in V$, the four-valued fuzzy graph assigns the following membership functions:

$$\mu_V(v), \lambda_V(v), \omega_V(v), \nu_V(v) : V \to [0, 1],$$

where:

- $\mu_V(v)$: Strong true membership degree of the vertex v,
- $\lambda_V(v)$: Contingent true membership degree of v,
- $\omega_V(v)$: Contingent false membership degree of v,

- $\nu_V(v)$: Strong false membership degree of v.

These membership functions must satisfy the condition:

$$\mu_V(v) + \lambda_V(v) + \omega_V(v) + \nu_V(v) = 1 \quad \forall v \in V.$$

For each edge $e = (u, v) \in E$, the four-valued fuzzy graph assigns the following membership functions:

$$\mu_E(e), \lambda_E(e), \omega_E(e), \nu_E(e) : E \to [0, 1],$$

where:

- $\mu_E(e)$: Strong true membership degree of the edge e,
- $\lambda_E(e)$: Contingent true membership degree of e,
- $\omega_E(e)$: Contingent false membership degree of e,
- $\nu_E(e)$: Strong false membership degree of e.

These membership functions must satisfy the condition:

$$\mu_E(e) + \lambda_E(e) + \omega_E(e) + \nu_E(e) = 1 \quad \forall e \in E.$$

The four-valued fuzzy graph G^{FVF} is represented as:

$$G^{FVF} = (V^{FVF}, E^{FVF}),$$

where:

- $V^{FVF} = \{(v, \mu_V(v), \lambda_V(v), \omega_V(v), \nu_V(v)) : v \in V\}$ is the four-valued fuzzy vertex set.
- $E^{FVF} = \{(e, \mu_E(e), \lambda_E(e), \omega_E(e), \nu_E(e)) : e \in E\}$ is the four-valued fuzzy edge set.

Theorem 4.40. *A General Plithogenic Graph with $s = 4$ and $t = 1$ satisfies all the conditions required to transform it into a Four-Valued Fuzzy Graph.*

Proof. The proof can be constructed in a manner similar to the previous theorem. □

4.2.12 Relation for Vague graphs

Vague graphs are a concept that graphically represent vague sets, and various studies have been conducted on them[25, 101, 103, 369, 393]. They are particularly closely related to intuitionistic fuzzy graphs.

We consider about Graph class for Vague graphs.

Theorem 4.41. *The following are examples of related graph classes, including but not limited to:*

- *Edge irregular product vague graphs[271]*
- *Vague incidence graphs[169]*
- *Complex vague graphs[479]*
- *Irregular vague graphs[88]*
- *Neutrosophic vague graphs[224]*
- *Neutrosophic vague line graphs[221]*
- *Cubic Vague Graphs[454]*
- *Vague Influence Graphs[406]*
- *Neutrosophic Bipolar Vague Line Graph[151]*
- *Neutrosophic vague incidence graph[221]*
- *Neutrosophic Bipolar Vague Incidence Graph[71]*
- *Neutrosophic Vague Soft Graphs[222]*

Proof. Refer to each reference as needed. □

The following is clearly satisfied.

Theorem 4.42. *A General Plithogenic Graph with $s = 2$ and $t = 1$ satisfies all the conditions required to transform it into a Vague graphs.*

Proof. The proof can be constructed in a manner similar to the previous theorem. □

4.2.13 Relation for Multi-fuzzy graphs

We explore the relationships within multi-fuzzy graphs, which are defined as an extension of the concept of multi-fuzzy sets [59, 398, 398, 399]. Related concepts include n-dimensional fuzzy sets [403], multidimensional fuzzy sets [167, 280, 395], and probabilistic hesitant fuzzy sets [168, 235, 254]. The definition is provided as follows.

Definition 4.43. Let $G = (V, E)$ be an undirected graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A **multi-fuzzy graph** G^M is a graph in which each vertex and each edge has multiple membership degrees associated with it.

Formally, a multi-fuzzy graph $G^M = (V^M, E^M)$ is defined as follows:

- **Multi-Fuzzy Vertex Set** V^M:

$$V^M = \{(v, \mu_1(v), \mu_2(v), \ldots, \mu_k(v)) \mid v \in V\}$$

  where $\mu_i(v) \in [0, 1]$ represents the i-th membership degree of vertex v for $i = 1, 2, \ldots, k$. The integer k denotes the dimension of the multi-fuzzy graph.

- **Multi-Fuzzy Edge Set** E^M:

$$E^M = \{(e, \nu_1(e), \nu_2(e), \ldots, \nu_k(e)) \mid e \in E\}$$

  where $\nu_i(e) \in [0, 1]$ represents the i-th membership degree of edge e for $i = 1, 2, \ldots, k$.

Theorem 4.44. *Let integer k the dimension of the multi-fuzzy graph. A General Plithogenic Graph with $s = k$ and $t = 1$ satisfies all the conditions required to transform it into a Multi-Fuzzy graphs.*

Proof. A General Plithogenic Graph G^{GP} is defined as $G^{GP} = (PM, PN)$, where:

1. **Plithogenic Vertex Set** $PM = (M, l, Ml, adf, aCf)$:
   - $M \subseteq V$: Set of vertices.
   - $adf : M \times Ml \to [0, 1]^s$: Degree of Appurtenance Function (DAF) for vertices.
   - $aCf : Ml \times Ml \to [0, 1]^t$: Degree of Contradiction Function (DCF) for vertices.
2. **Plithogenic Edge Set** $PN = (N, m, Nm, bdf, bCf)$:
   - $N \subseteq E$: Set of edges.
   - $bdf : N \times Nm \to [0, 1]^s$: Degree of Appurtenance Function (DAF) for edges.
   - $bCf : Nm \times Nm \to [0, 1]^t$: Degree of Contradiction Function (DCF) for edges.

Since $t = 1$, the Degree of Contradiction Functions aCf and bCf output a single value, thus $aCf : Ml \times Ml \to [0, 1]$ and $bCf : Nm \times Nm \to [0, 1]$.

A multi-fuzzy graph $G^M = (V^M, E^M)$ is defined with:

- **Multi-Fuzzy Vertex Set** V^M:

$$V^M = \{(v, \mu_1(v), \mu_2(v), \ldots, \mu_k(v)) \mid v \in V\}$$

  where $\mu_i(v) \in [0, 1]$ represents the i-th membership degree of vertex v for $i = 1, 2, \ldots, k$.

- **Multi-Fuzzy Edge Set** E^M:

$$E^M = \{(e, \nu_1(e), \nu_2(e), \ldots, \nu_k(e)) \mid e \in E\}$$

  where $\nu_i(e) \in [0, 1]$ represents the i-th membership degree of edge e for $i = 1, 2, \ldots, k$.

Given that $s = k$ in the General Plithogenic Graph G^{GP}:

- For each vertex $v \in M$, the Degree of Appurtenance Function $adf(v, a) = (\mu_1(v), \mu_2(v), \ldots, \mu_k(v))$ corresponds directly to the multi-membership degrees of v in the multi-fuzzy graph G^M. Thus, we can map each $adf(v, a)$ to the multi-fuzzy membership values $\mu_1(v), \mu_2(v), \ldots, \mu_k(v)$ for the vertex v.
- Similarly, for each edge $e \in N$, the Degree of Appurtenance Function $bdf(e, b) = (\nu_1(e), \nu_2(e), \ldots, \nu_k(e))$ corresponds to the multi-membership degrees of e in the multi-fuzzy graph G^M. We map $bdf(e, b)$ to $\nu_1(e), \nu_2(e), \ldots, \nu_k(e)$ for edge e.

Since aCf and bCf are now single values due to $t = 1$, they do not affect the multiple membership values in the transformation process. Therefore, we have proven that a General Plithogenic Graph with $s = k$ and $t = 1$ can be transformed into a Multi-Fuzzy Graph. □

4.2.14 Relation for intuitionistic multi-fuzzy graph

We consider the relationships for intuitionistic multi-fuzzy graphs. T hey are defined as graphs that extend the concept of intuitionistic multi-fuzzy sets [134, 137, 138]. The definition is as follows.

Definition 4.45. Let $G = (V, E)$ be a graph where V is a finite set of vertices and $E \subseteq V \times V$ is a set of edges. An intuitionistic multi-fuzzy graph G_{IMF} is defined as a pair $G_{IMF} = (V_{IMF}, E_{IMF})$ where:

1. **Vertices Membership Function**: For each vertex $v \in V$, there exists an associated intuitionistic multi-fuzzy set
$$\mu_G(v) = \{(\mu_1(v), \nu_1(v)), (\mu_2(v), \nu_2(v)), \ldots, (\mu_k(v), \nu_k(v))\},$$
where $0 \leq \mu_i(v) + \nu_i(v) \leq 1$ for each $i = 1, 2, \ldots, k$. Here, $\mu_i(v)$ represents the degree of membership of the vertex v at level i, and $\nu_i(v)$ represents the degree of non-membership of the vertex v at level i.

2. **Edges Membership Function**: Similarly, for each edge $e = (u, v) \in E$, there exists an associated intuitionistic multi-fuzzy set
$$\mu_G(e) = \{(\mu_1(e), \nu_1(e)), (\mu_2(e), \nu_2(e)), \ldots, (\mu_k(e), \nu_k(e))\},$$
where $0 \leq \mu_i(e) + \nu_i(e) \leq 1$ for each $i = 1, 2, \ldots, k$. Here, $\mu_i(e)$ represents the degree of membership of the edge e at level i, and $\nu_i(e)$ represents the degree of non-membership of the edge e at level i.

3. **Intuitionistic Multi-Fuzzy Value**: For each level i, the intuitionistic multi-fuzzy value (IMFV) or intuitionistic multi-fuzzy number (IMFN) is given by
$$\pi_i(v) = 1 - (\mu_i(v) + \nu_i(v))$$
for vertices, and
$$\pi_i(e) = 1 - (\mu_i(e) + \nu_i(e))$$
for edges. This value represents the degree of indeterminacy or hesitation at each level i.

The parameter k represents the number of different levels of membership and non-membership, allowing for a multi-faceted evaluation of each vertex and edge within the graph.

Theorem 4.46. *Let integer k the dimension of the multi-fuzzy graph. A General Plithogenic Graph with $s = 2\times k$ and $t = 1$ satisfies all the conditions required to transform it into a intuitionistic Multi-Fuzzy graphs.*

Proof. The proof can be constructed in a manner similar to the previous theorem. □

4.2.15 Relation for other concepts

Additionally, concepts such as the Hexapartitioned Neutrosophic Graph, which includes an additional membership function for hesitancy alongside truth, contradiction, ignorance, unknown, and falsity, can also be considered. These can similarly be generalized as Plithogenic graphs. Moreover, this generalization can be extended to Plithogenic sets rather than just Plithogenic graphs.

Question 4.47. Is it possible to define a Layered Plithogenic Graph? If so, how can it be defined?

4.3 Restricted Refined Plithogenic graphs

As previously mentioned, research on refined sets and graphs has advanced in recent years (cf.[146, 147, 238, 463, 463]). The definition of Restricted Refined Plithogenic Graphs is provided as follows.

Definition 4.48. Let $G = (V, E)$ be a crisp graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges.

We define a **Restricted Refined Plithogenic Graph (RRPG)** as:

$$RRPG = (RPM, RPN)$$

where:

- **Restricted Refined Plithogenic Vertex Set** $RPM = (M, l, Ml, adf_r, aCf_r, s, r)$:
  - $M \subseteq V$ is the set of vertices.
  - l is an attribute associated with the vertices.

- Ml is the range of possible attribute values.
- $adf_r : M \times Ml \to [0,1]^{s \times r}$ is the **Restricted Refined Degree of Appurtenance Function (RDAF)** for vertices.
- $aCf_r : Ml \times Ml \to [0,1]^{t \times r}$ is the **Restricted Refined Degree of Contradiction Function (RDCF)** for vertices.
- s is the number of uncertainty components (e.g., truth, indeterminacy, falsity).
- r is the number of splits per uncertainty component.

- **Restricted Refined Plithogenic Edge Set** $RPN = (N, m, Nm, bdf_r, bCf_r, s, r)$:
  - $N \subseteq E$ is the set of edges.
  - m is an attribute associated with the edges.
  - Nm is the range of possible attribute values.
  - $bdf_r : N \times Nm \to [0,1]^{s \times r}$ is the **Restricted Refined Degree of Appurtenance Function (RDAF)** for edges.
  - $bCf_r : Nm \times Nm \to [0,1]^{t \times r}$ is the **Restricted Refined Degree of Contradiction Function (RDCF)** for edges.
  - s and r are as defined above.

Theorem 4.49. *If $r = 1$ (i.e., no refinement is applied), the Restricted Refined Plithogenic Graph reduces to the classical Plithogenic Graph.*

Proof. When $r = 1$, each uncertainty component has only one degree, so the refined functions become:

$$adf_r : M \times Ml \to [0,1]^s$$
$$aCf_r : Ml \times Ml \to [0,1]^t$$

This matches the definitions of the classical Plithogenic Graph, where the Degree of Appurtenance Function and Degree of Contradiction Function output vectors of length s and t, respectively. All constraints and properties remain valid, thus confirming that the Restricted Refined Plithogenic Graph reduces to the classical case when $r = 1$. □

Theorem 4.50. *Under appropriate choices of s and t, the Restricted Refined Plithogenic Graph reduces to the corresponding restricted refined fuzzy, intuitionistic fuzzy, neutrosophic, Turiyam neutrosophic, or extended Turiyam Neutrosophic graphs.*

Proof. By selecting specific values for s and t, we can align the uncertainty components with those of the desired restricted refined graph:

- **Restricted Refined Fuzzy Graph**: Set $s = 1, t = 1, r \geq 1$.
- **Restricted Refined Intuitionistic Fuzzy Graph**: Set $s = 2, t = 1, r \geq 1$.
- **Restricted Refined Neutrosophic Graph**: Set $s = 3, t = 1, r \geq 1$.
- **Restricted Refined Turiyam Neutrosophic Graph**: Set $s = 4, t = 1, r \geq 1$.
- **Restricted Refined Extended Turiyam Neutrosophic Graph**: Set $s = 5, t = 1, r \geq 1$.

With these settings, the Restricted Refined Plithogenic Graph's definitions and constraints match those of the corresponding restricted refined graphs, confirming the reduction. □

Based on the above, the following relationship holds as illustrated in the diagram below. Note that in the diagram, "Turiyam Graph" refers to the "Turiyam Neutrosophic Graph."

Question 4.51. How is the definition of the Refined Plithogenic Graph related to other classes of Refined Graphs?

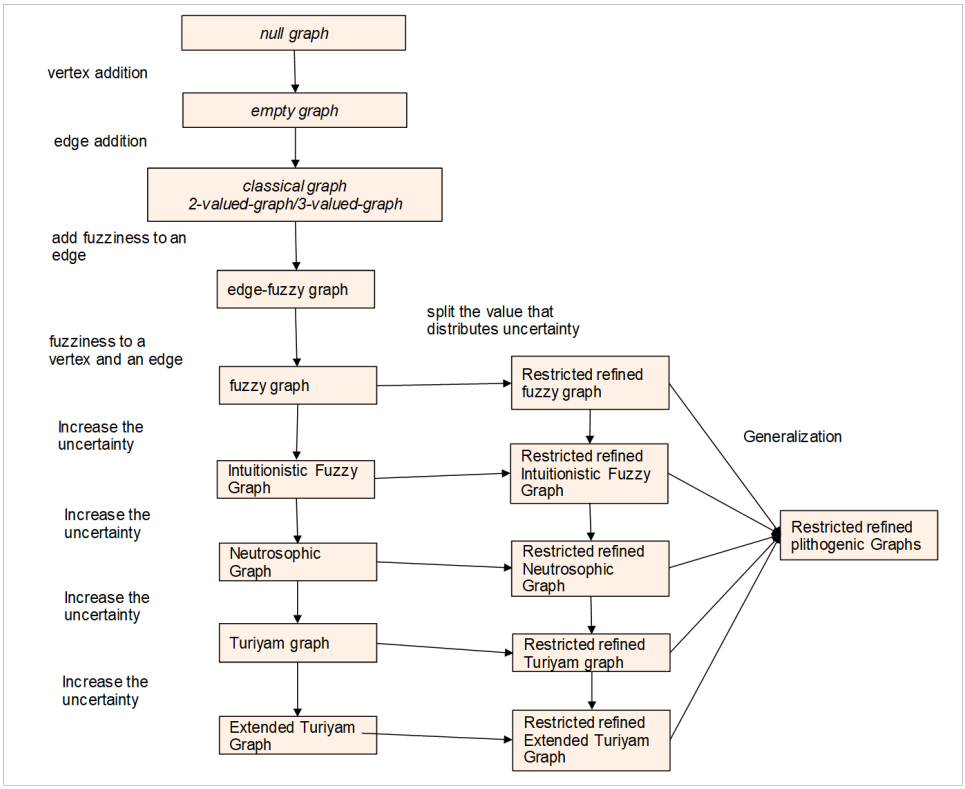

Fig. 2. Graph Hierarchy for Refined graph

4.4 Plithogenic Graph Type

Regarding the Plithogenic Graph type, specifically, we extend and examine the fuzzy graph type [97] into the Plithogenic Graph Type. As previously mentioned, I believe that determining where to incorporate uncertainty and where to maintain fixed elements is crucial for practical applications.

Definition 4.52 (Plithogenic Graph Type)**.** A plithogenic graph PG is a graph that satisfies one of the following types of plithogenic characteristics (referred to as PG of the i-th type) or any combination thereof:

(i) $PG_1 = \{G_1, G_2, G_3, \ldots, G_P\}$ where plithogenic characteristics exist in each graph G_i, incorporating different attributes and degrees of appurtenance and contradiction for vertices and edges.

(ii) $PG_2 = \{V, E_P\}$ where the edge set E_P is plithogenic, meaning that each edge is associated with a range of possible attributes and corresponding degrees of appurtenance and contradiction.

(iii) $PG_3 = \{V, E(t_P, h_P)\}$ where both the vertex set V and edge set E are crisp, but the edges have plithogenic heads $h(e_i)$ and plithogenic tails $t(e_i)$ with respect to certain attributes.

(iv) $PG_4 = \{V_P, E\}$ where the vertex set V_P is plithogenic, meaning each vertex has attributes with varying degrees of appurtenance and contradiction.

(v) $PG_5 = \{V, E(w_P)\}$ where both the vertex set V and edge set E are crisp, but the edges have plithogenic weights w_P, indicating the attribute-based degrees of appurtenance and contradiction.

The above corresponds to each graph type, specifically the fuzzy graph type, intuitionistic fuzzy graph type, neutrosophic graph type, Turiyam Neutrosophic graph type, and Extended Turiyam Neutrosophic graph type. It may be important to consider, apply, and study the appropriate graph type depending on the data being handled or the field of application.

Example 4.53. For the plithogenic graph PG, the values s and t correspond to the following graph types, each representing a specific plithogenic graph type:

- When $s = t = 1$, PG is classified as a **Plithogenic Fuzzy Graph**, denoted by PFG. This type corresponds to a fuzzy graph type where the degrees of appurtenance and contradiction for vertices and edges are governed by fuzzy logic, extending the concept of a traditional fuzzy graph into the plithogenic domain.
- When $s = 2$ and $t = 1$, PG becomes a **Plithogenic Intuitionistic Fuzzy Graph**, denoted by $PIFG$. As an intuitionistic fuzzy graph type, this form includes both membership and non-membership degrees for vertices and edges, thereby enhancing the traditional fuzzy graph with an intuitionistic dimension.
- When $s = 3$ and $t = 1$, PG is identified as a **Plithogenic Neutrosophic Graph**, denoted by PNG. This type, as a neutrosophic graph type, integrates three distinct attributes—degrees of truth, indeterminacy, and falsity—providing a more detailed and flexible representation within the plithogenic framework.
- When $s = 4$ and $t = 1$, PG is termed a **Plithogenic Turiyam Neutrosophic Graph**, denoted by $PTuG$. This graph type introduces four levels of evaluation for each attribute, adding an extra state beyond the neutrosophic type, thereby forming a more intricate type within the plithogenic classification.
- When $s = 5$ and $t = 1$, PG is known as a **Plithogenic Extended Turiyam Neutrosophic Graph**, denoted by $PETuG$. This represents the most comprehensive graph type, encompassing five different levels of evaluation for each attribute, and stands as the most generalized and inclusive plithogenic graph type.

5. Conclusions

In this paper, we introduced and explored a new class of graphs for Intuitionistic Fuzzy Graphs and Turiyam Neutrosophic Graphs, namely General Intuitionistic Fuzzy Graphs, General Turiyam Neutrosophic Graphs, and Turiyam Neutrosophic Hypergraphs. We also considered the concept of Extended Turiyam Neutrosophic Graphs. Furthermore, we investigated the potential for generalizing these graph types within the broader framework of Plithogenic Graphs. These concepts are not limited to graph theory alone; they can also be applied and hold true in other areas, such as set theory.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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